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Conway's ZIP Proof

George K. Francis and Jeffrey R. Weeks

Surfaces arise naturally in many different forms, in branches of mathematics ranging from complex analysis to dynamical systems. The Classification Theorem, known since the 1860's, asserts that all closed surfaces, despite their diverse origins and seemingly diverse forms, are topologically equivalent to spheres with some number of handles or crosscaps. The proofs found in most modern textbooks follow that of Seifert and Threlfall [**5**]. Seifert and Threlfall's proof, while satisfyingly constructive, requires that a given surface be brought into a somewhat artificial standard form. Here we present a completely new proof, discovered by John H. Conway in about 1992, which retains the constructive nature of [**5**] while eliminating the irrelevancies of the standard form. Conway calls it his Zero Irrelevancy Proof, or "ZIP proof," and asks that it always be called by this name, remarking that "otherwise there's a real danger that its origin would be lost, since everyone who hears it immediately regards it as the obvious proof." We trust that Conway's ingenious proof will replace the customary textbook repetition of Seifert-Threlfall in favor of a lighter, fat-free *nouvelle cuisine* approach that retains all the classical flavor of elementary topology.

We work in the realm of topology, where surfaces may be freely stretched and deformed. For example, a sphere and an ellipsoid are topologically equivalent, because one may be smoothly deformed into the other. But a sphere and a doughnut surface are topologically different, because no such deformation is possible. All of our figures depict deformations of surfaces. For example, the square with two holes in Figure 1A is topologically equivalent to the square with two tubes (1B), because one may be deformed to the other. More generally, two surfaces are considered equivalent, or *homeomorphic*, if and only if one may be

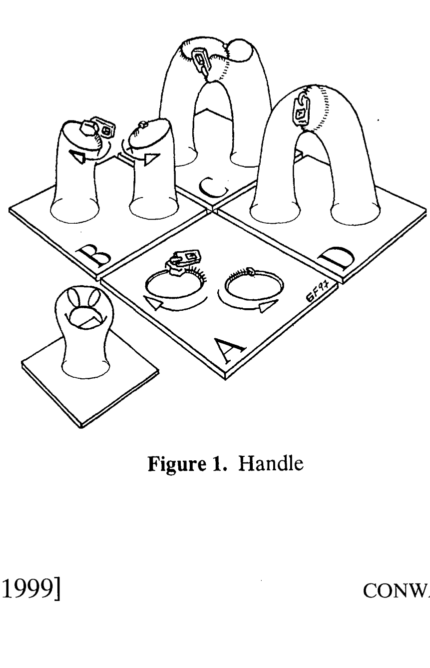

Figure 1. Handle

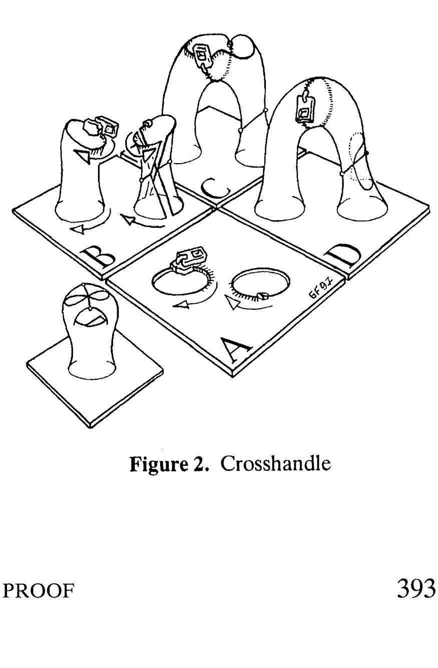

Figure 2. Crosshandle

mapped onto the other in a continuous, one-to-one fashion. That is, it's the final equivalence that counts, whether or not it was obtained via a deformation.

Let us introduce the primitive topological features in terms of zippers or "zip-pairs," a zip being half a zipper. Figure 1A shows a surface with two boundary circles, each with a zip. Zip the zips, and the surface acquires a *handle* (1D). If we reverse the direction of one of the zips (2A), then one of the tubes must "pass through itself" (2B) to get the zip orientations to match. Figure 2B shows the self-intersecting tube with a vertical slice temporarily removed, so the reader may see its structure more easily. Zipping the zips (2C) yields a *cross handle* (2D). This picture of a crosshandle contains a line of self-intersection. The self-intersection is an interesting feature of the surface's placement in 3-dimensional space, but has no effect on the intrinsic topology of the surface itself.

If the zips occupy two halves of a single boundary circle (Figure 3A), and their orientations are consistent, then we get a *cap* (3C), which is topologically trivial (3D) and won't be considered further. If the zip orientations are inconsistent (4A), the result is more interesting. We deform the surface so that corresponding points on the two zips lie opposite one another (4B), and begin zipping. At first

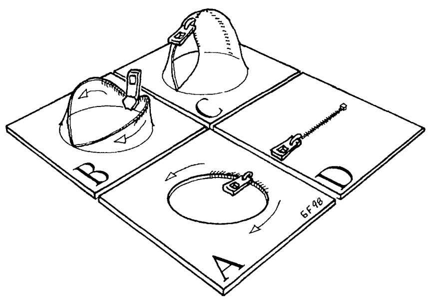

Figure 3. Cap

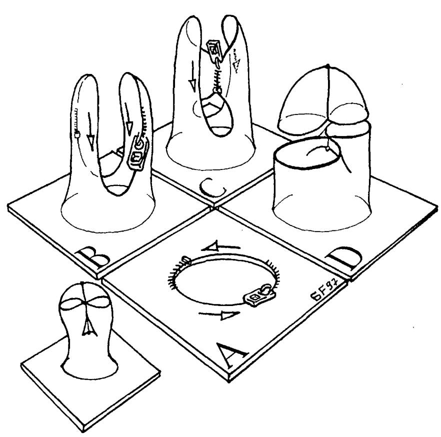

Figure 4. Crosscap

the zipper head moves uneventfully upward (4C), but upon reaching the top it starts downward, zipping together the "other two sheets" and creating a line of self-intersection. As before, the self-intersection is merely an artifact of the model, and has no effect on the intrinsic topology of the surface. The result is a *crosscap* (4D), shown here with a cut-away view to make the self-intersections clearer.

The preceding constructions should make the concept of a surface clear to non-specialists. Specialists may note that our surfaces are compact, and may have boundary.

Comment. A surface is *not* assumed to be connected.

Comment. Figure 5 shows an example of a triangulated surface. All surfaces may be triangulated, but the proof **[4]** is difficult. Instead we may consider the Classification Theorem to be a statement about surfaces that have already been triangulated.

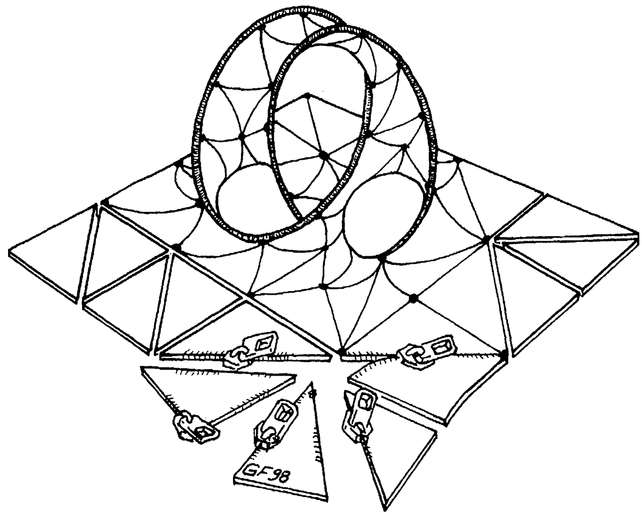

Figure 5. Install a zip-pair along each edge of the triangulation, unzip them all, and then re-zip them one at a time.

Definition. A *perforation* is what's left when you remove an open disk from a surface. For example, Figure 1A shows a portion of a surface with two perforations.

Definition. A surface is *ordinary* if it is homeomorphic to a finite collection of spheres, each with a finite number of handles, crosshandles, crosscaps, and perforations.

Classification Theorem (preliminary version). *Every surface is ordinary.*

Proof: Begin with an arbitrary triangulated surface. Imagine it as a patchwork quilt, only instead of imagining traditional square patches of material held together with stitching, imagine triangular patches held together with zip-pairs (Figure 5). Unzip all the zip-pairs, and the surface falls into a collection of triangles with zips along their edges. This collection of triangles is an ordinary surface, because each triangle is homeomorphic to a sphere with a single perforation. Now re-zip one zip to its original mate. It's not hard to show that the resulting surface must again be ordinary, but for clarity we postpone the details to Lemma 1. Continue re-zipping the zips to their original mates, one pair at a time, noting that at each step Lemma 1 ensures that the surface remains ordinary. When the last zip-pair is zipped, the original surface is restored, and is seen to be ordinary. ■

Lemma 1. *Consider a surface with two zips attached to portions of its boundary. If the surface is ordinary before the zips are zipped together, it is ordinary afterwards as well.*

Proof: First consider the case that each of the two zips completely occupies a boundary circle. If the two boundary circles lie on the same connected component of the surface, then the surface may be deformed so that the boundary circles are adjacent to one another, and zipping them together converts them into either a handle (Figure 1) or a crosshandle (Figure 2), according to their relative orientation. If the two boundary circles lie on different connected components, then zipping them together joins the two components into one.

Next consider the case that the two zips share a single boundary circle, which they occupy completely. Zipping them together creates either a cap (Figure 3) or a crosscap (Figure 4), according to their relative orientation.

Finally, consider the various cases in which the zips needn't completely occupy their boundary circle(s), but may leave gaps. For example, zipping together the zips in Figure 6A converts two perforations into a handle with a perforation on top (6B). The perforation may then be slid free of the handle (6C, 6D). Returning to the general case of two zips that needn't completely occupy their boundary circle(s), imagine that those two zips retain their normal size, while all other zips shrink to a size so small that we can't see them with our eyeglasses off. This reduces us (with our eyeglasses still off!) to the case of two zips that *do* completely occupy their boundary circle(s), so we zip them and obtain a handle, crosshandle, cap, or crosscap, as illustrated in Figures 1–4. When we put our eyeglasses back on, we notice that the surface has small perforations as well, which we now restore to their original size. ∎

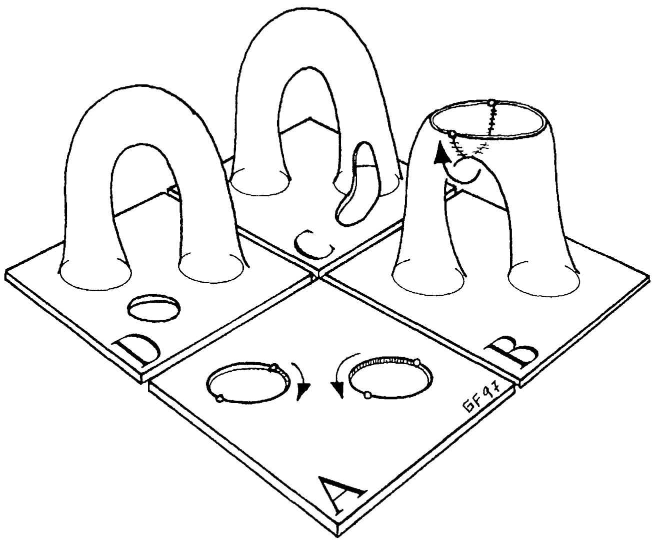

Figure 6. These zips only partially occupy the boundary circles, so zipping them yields a handle with a puncture.

The following two lemmas express the relationships among handles, crosshandles, and crosscaps.

Lemma 2. *A crosshandle is homeomorphic to two crosscaps.*

Proof: Consider a surface with a "Klein perforation" (Figure 7A). If the parallel zips (shown with black arrows in 7A) are zipped first, the perforation splits in two (7B). Zipping the remaining zips yields a crosshandle (7C).

If, on the other hand, the antiparallel zips (shown with white arrows in Figure 7A) are zipped first, we get a perforation with a "Möbius bridge" (7D). Raising its boundary to a constant height, while letting the surface droop below it, yields the bottom half of a crosscap (7E). Temporarily fill in the top half of the crosscap with an "invisible disk" (7F), slide the disk free of the crosscap's line of

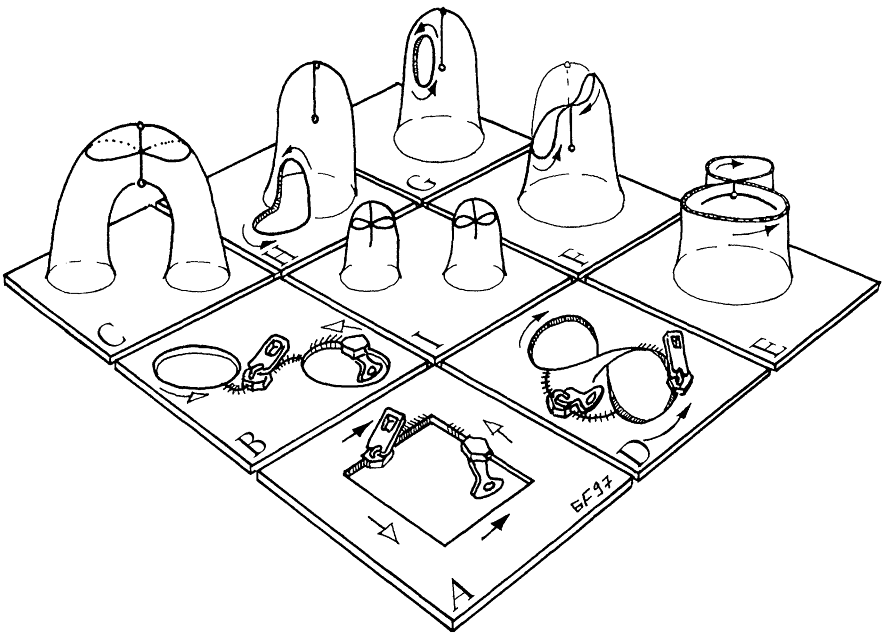

Figure 7. A crosshandle is homeomorphic to two crosscaps.

self-intersection (7G), and then remove the temporary disk. Slide the perforation off the crosscap (7H) and zip the remaining zip-pair (shown with black arrows) to create a second crosscap (7I).

The intrinsic topology of the surface does not depend on which zip-pair is zipped first, so we conclude that the crosshandle (7C) is homeomorphic to two crosscaps (7I). ■

Lemma 3 (Dyck's Theorem [1]). *Handles and crosshandles are equivalent in the presence of a crosscap.*

Proof: Consider a pair of perforations with zips installed as in Figure 8A. If, on the one hand, the black arrows are zipped first (8B), we get a handle along with instructions for a crosscap. If, on the other hand, one tube crosses through itself (8C, recall also Figure 2B) and the white arrows are zipped first, we get a crosshandle with instructions for a crosscap (8D). In both cases, of course, the crosscap may be slid free of the handle or crosshandle, just as the perforation was slid free of the handle in Figure 6BCD. Thus a handle-with-crosscap is homeomorphic to a crosshandle-with-crosscap. ■

Classification Theorem. *Every connected closed surface is homeomorphic to either a sphere with crosscaps or a sphere with handles.*

Proof: By the preliminary version of the Classification Theorem, a connected closed surface is homeomorphic to a sphere with handles, crosshandles, and crosscaps.

Case 1: At least one crosshandle or crosscap is present. Each crosshandle is homeomorphic to two crosscaps (Lemma 2), so the surface as a whole is homeomorphic to a sphere with crosscaps and handles only. At least one crosscap

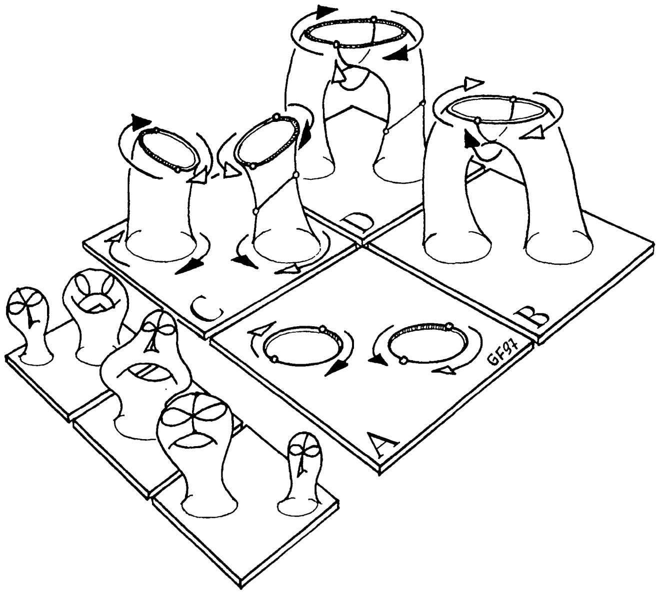

Figure 8. The presence of a crosscap makes a handle cross.

is present, so each handle is equivalent to a crosshandle (Lemma 3), which is in turn homeomorphic to two crosscaps (Lemma 2), resulting in a sphere with crosscaps only.

Case 2: No crosshandle or crosscap is present. The surface is homeomorphic to a sphere with handles only.

We have shown that every connected closed surface is homeomorphic to either a sphere with crosscaps or a sphere with handles. ■

Comment. The surfaces named in the Classification Theorem are all topologically distinct, and may be recognized by their orientability and Euler number. A sphere with n handles is orientable with Euler number $2 - 2n$, while a sphere with n crosscaps is nonorientable with Euler number $2 - n$. Most topology books provide details; elementary introductions appear in [**6**] and [**2**].

Nomenclature. A sphere with one handle is a *torus*, a sphere with two handles is a *double torus*, with three handles a *triple torus*, and so on. A sphere with one crosscap has traditionally been called a real projective plane. That name is appropriate in the study of projective geometry, when an affine structure is present, but is inappropriate for a purely topological object. Instead, Conway proposes that a sphere with one crosscap be called a *cross surface*. The name cross surface evokes not only the crosscap, but also the surface's elegant alternative construction as a sphere with antipodal points identified. A sphere with two crosscaps then becomes a *double cross surface*, with three crosscaps a *triple cross surface*, and so on. As special cases, the double cross surface is often called a *Klein bottle*, and the triple cross surface is often called *Dyck's surface* [**3**].

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A Problem From the MONTHLY 100 Years Ago

114. Proposed by F. P. MATZ, M.Sc., Ph.D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

Does it pay a \$4-carpenter using a dozen four-penny nails per minute, to pick up a dropped nail? At this rate, should twenty penny nails be picked up?

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Editors note: In the printed solution to the problem (by the MONTHLY's editor, Benjamin Finkel) one discovers that the carpenter was paid \$4 per *day* and that four-penny nails cost 5 cents per pound.

Chaos, Cantor Sets, and Hyperbolicity for the Logistic Maps

Roger L. Kraft

The family of logistic maps $f_\mu(x) = \mu x(1-x)$ appears in almost every dynamical systems textbook. It is one of the simplest nonlinear systems that one can study, but it is amazingly rich in phenomena. It has a surprising number of connections to other topics in dynamical systems and applied mathematics, for example, population dynamics, symbolic dynamics, complex analytic dynamics, the Mandelbrot set, the period-doubling route to chaos, renormalization, universality, homoclinic bifurcations, horseshoes, and invariant measures. Because of its simplicity, many introductory dynamical systems textbooks use it as a primary example, in particular as the primary example of a chaotic dynamical system. When $\mu > 2 + \sqrt{5} \approx 4.236$, it is not too difficult to prove that f_μ is chaotic on an invariant Cantor set; for the details, see any of [**1**, pp. 31–50], [**3**, pp. 112–126], or [**4**, pp. 69–85]. Each of these books states without proof that f_μ is actually chaotic for all $\mu > 4$. Our goal is to give a simple proof of this fact.

As far as I know, only one textbook gives a proof that f_μ is chaotic for $\mu > 4$ [**6**, pp. 33–37]. However, its proof uses the Poincaré hyperbolic metric on the unit interval, the calculation of a derivative using different metrics, and the Schwarz Lemma from the theory of complex variables. While this proof is very elegant, and hints at the connections between the logistic maps and complex analytic dynamics, it is not in the spirit of the more elementary books.

The family of logistic maps $f_\mu : \mathbb{R} \to \mathbb{R}$, $\mu > 0$, is a family of parabolas that open downward, intercept the x-axis at 0 and 1, and have a maximum at $1/2$. Since the maximum value is $\mu/4$, f_μ maps the interval $[0,1]$ into $[0,1]$ when $0 < \mu \leq 4$. But when $\mu > 4$, there are points in $[0,1]$ that escape from $[0,1]$ under forward iteration of f_μ. Let

$$\Lambda_\mu \equiv \bigcap_{n=1}^{\infty} f_\mu^{-n}([0,1]).$$

For $\mu > 4$, Λ_μ contains exactly those points in $[0,1]$ that never escape under forward iteration by f_μ. Our main result is:

Theorem 1. *If $\mu > 4$, then Λ_μ is a Cantor set, and the restriction of f_μ to Λ_μ is chaotic.*

Once we have shown that Λ_μ is a Cantor set, the proof that the restriction of f_μ to Λ_μ is chaotic is same as in the case $\mu > 2 + \sqrt{5}$: Use itineraries to construct a topological conjugacy between f_μ on Λ_μ and the shift map σ on $\Sigma_2 = \{0,1\}^{\mathbb{N}}$; this shows that f_μ on Λ_μ is topologically transitive and has dense periodic points. It is easy to show that f_μ on Λ_μ has sensitive dependence on initial conditions; in fact, it is easy to show that it is expansive, which is a stronger property [**1**, p. 50], [**6**, p. 83]. All the details of these steps remain unchanged.

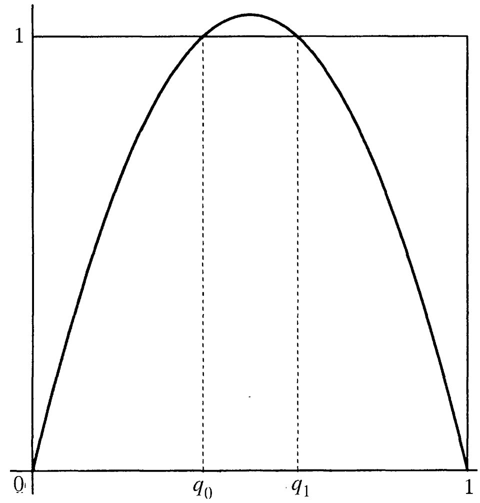

Figure 1.

Choose a value of $\mu > 4$, and let us define some notation. Let $q_0 < q_1$ solve $f_\mu(x) = 1$; see Figure 1. Let $I_0 = [0,1]$, and let $I_1 = [0, q_0] \cup [q_1, 1]$. Notice that $I_1 = f_\mu^{-1}(I_0)$. In general, let $I_n = f_\mu^{-1}(I_{n-1}) = f_\mu^{-n}(I_0)$. Then I_n is exactly those points in $[0,1]$ that stay in $[0,1]$ for their first n iterates under f_μ, and $I_n \subset I_{n-1}$. Notice also that I_n is made up of 2^n disjoint closed intervals. Order the 2^n components of I_n from left to right, and let $I_{n,j}$ denote the j-th component. (To keep the notation as simple as possible, we suppress the explicit dependence of objects like I_n on the parameter μ.) For more details about the definitions in this paragraph, see [**1**, pp. 34–36], [**3**, pp. 112–114], [**4**, pp. 70–73], or [**6**, pp. 30–32].

If I is an interval, we let $|I|$ denote the length of I.

Recall that a subset of the real line is a *Cantor set* if it is compact, perfect, and totally disconnected. Recall also that a subset of the real line is *totally disconnected* if and only if it does not contain any intervals; see [**1**, p. 37], [**3**, p. 116], [**4**, p. 73], or [**6**, p. 26].

The first step in proving that Λ_μ is a Cantor set is the following lemma.

Lemma 2. *If* $\mu > 4$, *then* Λ_μ *is a compact perfect set.*

Proof: Since $\Lambda_\mu = \cap_{n=0}^\infty I_n$ and each I_n is compact, we know that Λ_μ is compact. To show that Λ_μ is perfect, first notice that for every n, all the endpoints of I_n are contained in Λ_μ. Let $x \in \Lambda_\mu$, and for each n let I_{n,j_x} denote the component of I_n that contains x. If $|I_{n,j_x}| \to 0$ as $n \to \infty$, then there are endpoints from I_{n,j_x} arbitrarily close to x, so x is in the closure of $\Lambda_\mu \setminus \{x\}$. On the other hand, if $|I_{n,j_x}|$ does not go to 0 as $n \to \infty$, then $\cap_{n=0}^\infty I_{n,j_x}$ is a closed interval, and $x \in \cap_{n=0}^\infty I_{n,j_x} \subset \Lambda_\mu$, so once again x is in the closure of $\Lambda_\mu \setminus \{x\}$. This shows that Λ_μ is perfect. ∎

To finish the proof that Λ_μ is a Cantor set, we need to show that it does not contain any intervals. How is this done when $\mu > 2 + \sqrt{5}$? A simple calculation shows that when $\mu = 2 + \sqrt{5}$, $f_\mu'(q_0) = 1$. So if $\mu > 2 + \sqrt{5}$, then $|f_\mu'(x)| > 1$ for all $x \in I_1$. This key fact makes the case $\mu > 2 + \sqrt{5}$ straightforward, as we now show.

Lemma 3. *If $\mu > 2 + \sqrt{5}$, then Λ_μ is a Cantor set.*

Proof: Suppose that Λ_μ contains an interval; let $[a, b] \subset \Lambda_\mu$. For every $n \geq 1$, the Mean Value Theorem applied to f_μ^n on the interval $[a, b]$ ensures that there is a point $c_n \in (a, b)$ such that

$$f_\mu^n(b) - f_\mu^n(a) = \left(f_\mu^n\right)'(c_n)(b - a).$$

Let $\lambda = f_\mu'(q_0)$, so $|f_\mu'(x)| \geq \lambda$ for all $x \in I_1$. Since $\mu > 2 + \sqrt{5}$, we have $\lambda > 1$. Since $c_n \in [a, b] \subset \Lambda_\mu$, we have $f_\mu^i(c_n) \in \Lambda_\mu \subset I_1$ for all $0 \leq \mathrm{i} \leq \mathrm{n} - 1$. Therefore, $|(f_\mu^n)'(c_n)| \geq \lambda^n$ by the chain rule, and

$$\left|f_\mu^n(b) - f_\mu^n(a)\right| = \left|\left(f_\mu^n\right)'(c_n)\right| \cdot |b - a| \geq \lambda^n |b - a|.$$

Since $\lambda > 1$, this implies that $|f_\mu^n(b) - f_\mu^n(a)| > 1$ for all n sufficiently large. But Λ_μ is invariant, and hence $\{f^n(a), f^n(b)\} \subset \Lambda_\mu \subset [0, 1]$ for all n, so we have a contradiction. Thus Λ_μ does not contain any intervals, and hence is a Cantor set. ∎

Here is another way to think about this proof: If $\mu > 2 + \sqrt{5}$, and we apply f_μ^{-1} to I_n to get I_{n+1}, then f_μ^{-1} shrinks the length of every component of I_n by at least the amount $\lambda^{-1} < 1$, so the lengths of the components of I_n go to zero as n goes to infinity.

When $\mu \in (4, 2 + \sqrt{5}]$, we have $|f_\mu'(x)| > 1$ for some $x \in I_1$, but $|f_\mu'(x)| \leq 1$ for other $x \in I_1$. When we apply f_μ^{-1} to I_n to get I_{n+1}, f_μ^{-1} shrinks some components of I_n, but, in contrast to the case when $\mu > 2 + \sqrt{5}$, f_μ^{-1} may also stretch other components of I_n. This combination of shrinking and stretching by f_μ^{-1} is what makes it difficult to show that Λ_μ is a Cantor set when $4 < \mu \leq 2 + \sqrt{5}$. However, a little playing around with f_μ should give one the sense that somehow, the stretching is eventually dominated by the shrinking as we repeatedly apply f_μ^{-1}. This leads to the following important definition.

Definition. Let $f: \mathbb{R} \to \mathbb{R}$ be a C^1 function, and suppose that Λ is a compact invariant set for f (i.e., $f(\Lambda) = \Lambda$). Then Λ is a *hyperbolic set* for f if there are constants $C > 0$ and $\lambda > 1$ such that $|(f^n)'(x)| \geq C\lambda^n$ for all $x \in \Lambda$ and all $n \geq 1$.

The C in the definition takes care of the fact that f^{-1} may stretch some intervals (i.e., $|f'(x)| \leq 1$ for some $x \in \Lambda$), in which case $C < 1$, but $\lambda > 1$ implies that shrinking under f^{-n} eventually dominates any stretching when $C\lambda^n > 1$; see [**6**, pp. 107–108 and p. 156].

The following lemma gives some insight into the definition of hyperbolicity, and makes it easier to use.

Lemma 4. *Let $f: \mathbb{R} \to \mathbb{R}$ be a C^1 function, and suppose that Λ is a compact invariant set for f. Then the following are equivalent.*

(1) *There are constants $C > 0$ and $\lambda > 1$ such that $|(f^n)'(x)| \geq C\lambda^n$ for all $x \in \Lambda$ and all $n \geq 1$.*
(2) *There is an integer $N \geq 1$ such that $|(f^n)'(x)| > 1$ for all $x \in \Lambda$ and all $n \geq N$.*
(3) *There is an integer $n_0 \geq 1$ such that $|(f^{n_0})'(x)| > 1$ for all $x \in \Lambda$.*
(4) *For every $x \in \Lambda$ there is an integer $n_x \geq 1$, which may depend on x, such that $|(f^{n_x})'(x)| > 1$.*

Remark. If $|f'(x)| > 1$ for all $x \in \Lambda$, then it is obvious that all four of the conditions in the lemma are true. This emphasizes once again that it is the possibility that $|f'(x)| \leq 1$ for some $x \in \Lambda$ that makes the definition of hyperbolicity subtle.

Proof: (4) $\Rightarrow$ (3) [**5**, p. 220] Since f is C^1, $(f^n)'$ is continuous for every n. For each $x \in \Lambda$, $|(f^{n_x})'(x)| > 1$ and $(f^{n_x})'$ is continuous, so there is neighborhood U_x of x and a $\lambda_x > 1$ such that $|(f^{n_x})'(y)| > \lambda_x$ for all $y \in U_x$. The open sets $\{U_x | x \in \Lambda\}$ cover the compact set Λ, so there is a finite subcover $\{U_i\}_{i=1}^k$, numbers $\{\lambda_i\}_{i=1}^k$ all strictly greater than 1, and integers $\{n_i\}_{i=1}^k$ such that $|(f^{n_i})'(y)| > \lambda_i$ for all $y \in U_i$. Let

$$\nu = \max\{n_1 \dots n_k\}, \quad \lambda_0 = \min\{\lambda_1 \dots \lambda_k\}, \quad \text{and} \quad m = \min_{x \in \Lambda}\{|f'(x)|\},$$

so $m > 0$ (why?). Choose an integer k so that $\lambda_0^k m^\nu > 1$, and let $n_0 = k\nu + \nu$. Now that we have defined our global choice for n_0, we need to show that $|(f^{n_0})'(x)| > 1$ for all $x \in \Lambda$. If we imagine that λ_0 represents "good" derivatives ($\lambda_0 > 1$) and m represents "bad" derivatives ($m < 1$), then we need to show that $f^{n_0}(x)$ contains at least k iterates with good derivatives to compensate for the worst case of ν iterates with bad derivatives.

Choose $x \in \Lambda$ and perform the following selection process that depends on x and terminates after a finite number of steps:

Choose ν_1 so that $x \in U_{\nu_1}$. Now suppose that we are given $\{\nu_1, \dots, \nu_j\}$. Let $\eta = \sum_{i=1}^j n_{\nu_i}$. Choose ν_{j+1} so that $f^\eta(x) \in U_{\nu_{j+1}}$. If $\eta + \nu_{j+1} > k\nu$, then stop; otherwise, go on to choose ν_{j+2}.

If the selection process stops after j steps, then $k\nu < \sum_{i=1}^j n_{\nu_i} \leq k\nu + \nu$. Write $n_0 = n_{\nu_1} + n_{\nu_2} + \cdots + n_{\nu_j} + i_x$, where $0 \leq i_x \leq \nu$. Each n_{ν_i} represents a good iterate (derivative > 1), j represents how many good iterates we actually have, and i_x represents how many bad iterates (derivatives < 1) we actually have. Since each $n_{\nu_i} \leq \nu$, we know that $j \geq k$. Using the chain rule, we can estimate $|(f^{n_0})'(x)|$:

$$\begin{aligned}
|(f^{n_0})'(x)| &= \left|\left(f^{i_x}(f^{n_{\nu_j}}(f^{n_{\nu_{j-1}}} \dots f^{n_{\nu_1}}))\right)'(x)\right| \\
&\geq m^{i_x}\lambda_{\nu_j}\lambda_{\nu_{j-1}} \dots \lambda_{\nu_1} && \left(\text{by the properties of the subcover } \{U_i\}_{i=1}^k\right) \\
&\geq m^\nu\lambda_{\nu_j}\lambda_{\nu_{j-1}} \dots \lambda_{\nu_1} && (\text{since } m \leq 1 \text{ and } i_x \leq \nu) \\
&\geq m^\nu\lambda_0^j && (\text{by our choice of } \lambda_0) \\
&\geq m^\nu\lambda_0^k && (\text{since } j \geq k \text{ and } \lambda_0 > 1) \\
&> 1 && (\text{because of our choice of } k).
\end{aligned}$$

Thus, $|(f^{n_0})'(x)| > 1$ for all $x \in \Lambda$, which proves (3).

(3) $\Rightarrow$ (2) [**1**, p. 99] If $n_0 = 1$ in (3), there is nothing to prove, so suppose $n_0 > 1$. Let

$$\lambda = \min_{x \in \Lambda}\{|(f^{n_0})'(x)|\} \quad \text{and} \quad m = \min_{x \in \Lambda}\{|f'(x)|\},$$

so $\lambda > 1$ (why?) and $m > 0$. Since we are assuming that $n_0 > 1$, we must have $m \leq 1$. Choose k so that $m^{n_0 - 1}\lambda^k > 1$. Let $N = n_0 k + (n_0 - 1)$. If $n > N$, write

$n = n_0(k + \nu) + i$, where $\nu > 0$ and $0 \le i \le n_0 - 1$. Then for any $x \in \Lambda$ we have

$$\begin{aligned}
|(f^n)'(x)| &= |(f^{n_0(k+\nu)})'(f^i(x))| \cdot |(f^i)'(x)| \\
&\ge \lambda^{k+\nu} m^i \\
&\ge \lambda^{\nu} \lambda^k m^{n_0-1} && (\text{since } m \le 1 \text{ and } i \le n_0 - 1) \\
&> \lambda^{\nu} && (\text{by our choice of } k) \\
&> 1 && (\text{since } \lambda > 1).
\end{aligned}$$

(2) $\Rightarrow$ (1) If $N = 1$ in (2), there is nothing to prove, so suppose $N > 1$. Let

$$m_1 = \min_{x \in \Lambda} \{ |(f^N)'(x)| \} \quad \text{and} \quad m = \min_{x \in \Lambda} \{ |f'(x)| \},$$

so $m_1 > 1$. Since we are assuming that $N > 1$, we must have $m \le 1$. Let

$$\lambda = m_1^{1/N} \quad \text{and} \quad C = (m/\lambda)^{N-1},$$

so $\lambda > 1$ and $C > 0$. For any $n > 0$ write $n = kN + i$, where $k \ge 0$ and $0 \le i \le N - 1$. Then for any $x \in \Lambda$ we have

$$\begin{aligned}
|(f^n)'(x)| &= |(f^{kN})'(f^i(x))| \cdot |(f^i)'(x)| \\
&\ge m_1^k m^i \\
&= \lambda^{kN} m^i && (\text{by our choice of } \lambda) \\
&= \lambda^{kN} \lambda^i (m/\lambda)^i \\
&\ge \lambda^{kN+i} (m/\lambda)^{N-1} && (\text{since } m/\lambda < 1 \text{ and } i \le N - 1) \\
&= C\lambda^n && (\text{by our choice of } C).
\end{aligned}$$

(1) $\Rightarrow$ (4) Choose n large enough so that $C\lambda^n > 1$. Then we have $|(f^n)'(x)| \ge C\lambda^n > 1$. Now let $n_x = n$ for every $x \in \Lambda$. ■

Why so many versions of the definition of hyperbolic? When we want to prove that a set is hyperbolic, it helps to use the weakest version of the definition, (4). On the other hand, when we want to prove general conclusions about a hyperbolic set, then it helps to use the strongest version, (1). Also, (2) is used as the definition of a hyperbolic set in some textbooks when the emphasis is on dynamics in one dimension, e.g., [**1**, p. 38], or [**4**, p. 77]. But a generalization of (1) is used in the definition of hyperbolicity for higher dimensions, e.g., [**6**, p. 241].

When $\mu > 2 + \sqrt{5}$, we have $|f_\mu'(x)| > 1$ for all $x \in I_1$, and this is the key to proving that Λ_μ is a Cantor set. To prove that Λ_μ is a Cantor set when $\mu > 4$, we need to replace "$|f_\mu'(x)| > 1$ for all $x \in I_1$" with "Λ_μ is a hyperbolic set for f_μ." Before we can begin the proof of hyperbolicity, we need to introduce an important tool, the Schwarzian derivative.

Definition. The *Schwarzian derivative* of a C^3 function f at a point x where $f'(x) \neq 0$ is

$$Sf(x) \equiv \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

This strange definition turns out to be tremendously useful. Our first result about the Schwarzian derivative is the following lemma.

Lemma 5. *For the logistic family f_μ with $\mu > 0$,*

(1) $Sf_\mu(x) < 0$ *for all* $x \in \mathbb{R} \setminus \{1/2\}$,
(2) $Sf_\mu^n(x) < 0$ *for all* $n > 1$ *and all* $x \in \mathbb{R} \setminus \cup_{i=0}^{n-1} f_\mu^{-i}(1/2)$.

The first item is easy, since $f_\mu''' \equiv 0$. However, the second is not so obvious, since f_μ^n is a polynomial of degree 2^n. The second item follows from the first item, the following lemma, and induction. This "hereditary" result is one of the reasons the Schwarzian derivative is so useful.

Lemma 6. *If $g'(x) \neq 0$ and $f'(g(x)) \neq 0$, then*

$$S(f \circ g)(x) = Sf(g(x)) \cdot (g'(x))^2 + Sg(x).$$

So if $Sg(x) < 0$ and $Sf(g(x)) < 0$, then $S(f \circ g)(x) < 0$.

Proof: The chain rule gives

$$\begin{aligned}
(f \circ g)'(x) &= f'(g(x))g'(x), \\
(f \circ g)''(x) &= f''(g(x))(g'(x))^2 + f'(g(x))g''(x), \quad \text{and} \\
(f \circ g)'''(x) &= f'''(g(x))(g'(x))^3 + 3f''(g(x))g''(x)g'(x) + f'(g(x))g'''(x).
\end{aligned}$$

A computation now gives the desired result. ■

We say that a function f has *negative Schwarzian derivative on an interval I* if $f'(x) \neq 0$ and $Sf(x) < 0$ for all $x \in I$; we abbreviate this as $Sf < 0$ on I. The following lemma gives a geometric consequence of negative Schwarzian derivative.

Lemma 7. *If I is an open interval and $Sf < 0$ on I, then f' cannot have a positive local minimum on I, nor can it have a negative local maximum.*

Proof: Suppose that x is a positive local minimum point for f' on I. Then $f'(x) > 0$, $f''(x) = 0$, and $f'''(x) \geq 0$ (why?). This implies that $Sf(x) \geq 0$, which contradicts $Sf < 0$ on I.

Similarly, if x is a negative local maximum point for f' on I, then $f'(x) < 0$, $f''(x) = 0$, and $f'''(x) \leq 0$. This implies $Sf(x) \geq 0$, which contradicts $Sf < 0$ on I. ■

Lemma 8 (Minimum Principle). *Let $I = [a, b]$ and suppose f is C^3 on I. If $Sf < 0$ on (a, b), then $|f'(x)| > \min\{|f'(a)|, |f'(b)|\}$ for all $x \in (a, b)$.*

Proof: Since $|f'|$ is continuous on the closed interval I, it must have a minimum at some point $x_0 \in I$. If $x_0 \in (a, b)$, then $f'(x_0) \neq 0$ since $Sf < 0$ on (a, b). If $f'(x_0) > 0$, then f' has a positive local minimum on (a, b), which contradicts Lemma 7. On the other hand, if $f'(x_0) < 0$, then f' has a negative local maximum on (a, b), another contradiction of Lemma 7. It follows that $x_0 = a$ or $x_0 = b$. ■

The Minimum Principle is the key result we need about negative Schwarzian derivatives. When we apply it to the iterates f_μ^n, we get information about the

shape of the graph of f_μ^n between its critical points that would be very difficult to get in any other way.

There are other important consequences for one-dimensional dynamical systems of negative Schwarzian derivative; see [**1**, Section 1.11].

Now consider f_μ. For any $\mu > 0$, f_μ has a fixed point at $p_1 = 1 - (1/\mu)$, and $f'_\mu(p_1) = 2 - \mu$, so $|f'_\mu(p_1)| > 1$ when $\mu > 3$. Let $p_0 = 1/\mu$, so p_0 and p_1 are symmetric about $1/2$; see Figure 2. Notice that $f_\mu(p_0) = p_1$ (when $\mu > 2$), and that $f_\mu([p_0, q_0]) = f_\mu([q_1, p_1]) = [p_1, 1]$. So if we let $J = (p_0, q_0) \cup (q_1, p_1)$, and if x is any point in J, then $f_\mu(x) \notin J$. But we have the following "return lemma."

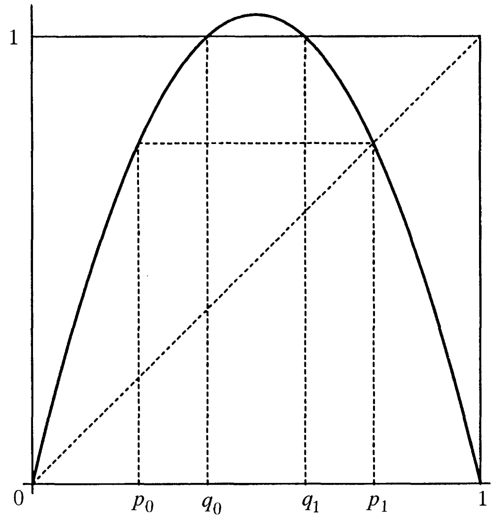

Figure 2.

Lemma 9 (Return Lemma). *If $\mu > 4$ and if $x \in J$, then there is an integer $n \geq 2$ such that $f_\mu^n(x) \in [p_0, p_1)$.*

Proof: Choose $x \in J$, so $f_\mu(x) \in (p_1, 1)$ and $f_\mu^2(x) \in (0, p_1)$. If $f_\mu^2(x) \in [p_0, p_1)$, then we are done. Suppose that $f_\mu^2(x) \in (0, p_0)$. We claim that for some $n \geq 1$, $f_\mu^{2+n}(x)$ is in $[p_0, p_1)$. Suppose not. Since $f_\mu(z) > z$ for all $z \in (0, p_0)$, we know that $f_\mu^{2+n}(x)$ is an increasing sequence bounded from above by p_0. So $f_\mu^{2+n}(x)$ converges as $n \to \infty$ to some point $z_0 \leq p_0$. It follows that z_0 is a fixed point for f_μ. But $0 < z_0 < p_1$, so we have a contradiction. ∎

Lemma 10. *If $\mu > 4$, then $q_0 - p_0 < p_0$. So the intervals (p_0, q_0) and (q_1, p_1) are shorter than the intervals $(0, p_0)$ and $(p_1, 1)$.*

Proof: We need to show that $\mu > 4$ implies $2p_0 > q_0$. Recall that

$$p_0 = \frac{1}{\mu} \quad \text{and} \quad q_0 = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{\mu}}.$$

Since $\mu > 4$, we have $0 < 1 - (4/\mu) < 1$, so $\sqrt{1 - (4/\mu)} > 1 - (4/\mu)$. After multiplying both sides by $1/2$, we have

$$\sqrt{\frac{1}{4} - \frac{1}{\mu}} > \frac{1}{2} - \frac{2}{\mu},$$

or

$$2\left(\frac{1}{\mu}\right) > \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{\mu}}. \qquad \blacksquare$$

Now we have all the ingredients we need to prove hyperbolicity.

Theorem 11. *If* $\mu > 4$, *then* Λ_μ *is a hyperbolic set for* f_μ.

Proof: Let $x \in \Lambda_\mu$. Assume that $x > 1/2$; the case where $x < 1/2$ follows by the symmetry of f_μ about $1/2$. We need to find an integer n (which may depend on x) such that $|(f_\mu^n)'(x)| > 1$. The hyperbolicity of Λ_μ then follows from Lemma 4.

If $x \geq p_1$, we can let $n = 1$ (why?). If $x = q_1$, then $f_\mu^n(q_1) = 0$ for $n \geq 2$, so

$$\left|(f_\mu^n)'(q_1)\right| = \left|f_\mu'(q_1)\right| \cdot \left|f_\mu'(1)\right| \cdot \left|f_\mu'(0)\right|^{n-2} = \mu^{n-1}\sqrt{\mu^2 - 4\mu} = \mu^n\sqrt{1 - (4/\mu)},$$

which is strictly greater than 1 for all n sufficiently large.

Now concentrate on x between q_1 and p_1. The Return Lemma ensures that there is an n such that $f_\mu^n(x) \in [p_0, p_1)$. Let $I_{n,j}$ be the component of I_n that contains x. There are two cases to consider: either $I_{n,j} \subset [q_1, p_1)$, or it is not.

Suppose $I_{n,j} \subset [q_1, p_1)$. Since f_μ^n maps $I_{n,j}$ monotonically onto $[0,1]$ (see [**1**, p. 36], [**4**, p. 71]), or [**6**, p. 31]), we can partition $I_{n,j}$ into three subintervals, $I_{n,j} = L_{n,j} \cup K_{n,j} \cup R_{n,j}$, where $f_\mu^n(L_{n,j}) = [0, p_0]$, $f_\mu^n(K_{n,j}) = (p_0, p_1)$, and $f_\mu^n(R_{n,j}) = [p_1, 1]$. Since $L_{n,j} \subset I_{n,j} \subset [q_1, p_1)$ and $R_{n,j} \subset I_{n,j} \subset [q_1, p_1)$, Lemma 10 ensures that $|f_\mu^n(L_{n,j})| > |L_{n,j}|$ and $|f_\mu^n(R_{n,j})| > |R_{n,j}|$. That is, f_μ^n must do some stretching near both ends of $I_{n,j}$. By the Mean Value Theorem applied to f_μ^n, there is a point $y \in L_{n,j}$ and a point $z \in R_{n,j}$ such that $|(f_\mu^n)'(y)| > 1$ and $|(f_\mu^n)'(z)| > 1$. Since $f_\mu^n(x) \in [p_0, p_1)$, we have $x \in \text{closure}(K_{n,j})$, so $y \leq x < z$. Since f_μ^n does not have a critical point in $[y, z]$, the Minimum Principle ensures that $|(f_\mu^n)'(x)| > 1$.

Now suppose that $I_{n,j}$ is not a subset of $[q_1, p_1)$. Once again, partition $I_{n,j}$ into three subintervals, $I_{n,j} = L_{n,j} \cup K_{n,j} \cup R_{n,j}$, where $f_\mu^n(L_{n,j}) = [0, p_0]$, $f_\mu^n(K_{n,j}) = (p_0, p_1)$, and $f_\mu^n(R_{n,j}) = [p_1, 1]$. As before, $x \in \text{closure}(K_{n,j})$ because $f_\mu^n(x) \in [p_0, p_1)$. Since $x \in (q_1, p_1)$, one of $L_{n,j}$ or $R_{n,j}$ is contained in $[q_1, p_1)$, but, since $I_{n,j}$ is not a subset of $[q_1, p_1)$, the other one of $L_{n,j}$ or $R_{n,j}$ is not contained in $[q_1, p_1)$. Suppose that $L_{n,j}$ is contained in $[q_1, p_1)$, and $R_{n,j}$ is not (the other case is similar). Since $I_{n,j} \subset [q_1, 1]$ and $I_{n,j} \cap [q_1, p_1) \neq \varnothing$, it must be that $p_1 \in I_{n,j}$. As before, $|f_\mu^n(L_{n,j})| > |L_{n,j}|$, so the Mean Value Theorem ensures that there is a point $y \in L_{n,j}$ such that $|(f_\mu^n)'(y)| > 1$. And $|(f_\mu^n)'(p_1)| > 1$ since p_1 is a hyperbolic repelling fixed point. Then $x \in [y, p_1]$ and f_μ^n does not have a critical point in $[y, p_1]$, so the Minimum Principle ensures that $|(f_\mu^n)'(x)| > 1$. $\blacksquare$

Remark. This proof of the hyperbolicity of Λ_μ is adapted from the idea of an "induced map" or "first return map" for f_μ; see [**2**, p. 341], and see [**1**, pp. 75–78] for an application of this idea when $\mu = 4$.

Theorem 12. *If $\mu > 4$, then Λ_μ is a Cantor set.*

Proof: The proof is now as easy as the case $\mu > 2 + \sqrt{5}$. Just observe that since $c_n \in \Lambda_\mu$, the hyperbolicity of Λ_μ ensures that $|(f_\mu^n)'(c_n)| \geq C\lambda^n$. The rest of the proof is unchanged. ■

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From the MONTHLY 100 Years Ago

The following are some of the advanced courses of Mathematics offered for the year 1899–1900 at the University of Chicago: Twisted Curves and Surfaces, Associate Professor Maschke; Projective Geometry, Professor Moore; Theory of Invariants, Professor Bolza; Continuous Groups, Professor Bolza; Theory of Functions of a Complex Variable, Professor Moore and Associate Professor Maschke; Elliptic Functions, Professor Bolza; Hyperelliptic Functions, Professor Bolza; Abstract Groups, Associate Professor Maschke; Elliptic Modular Functions, Professor Moore; Theory of Substitution, Professor Moore; Theory of Numbers, Assistant Professor Young, etc., etc.

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An Elementary View of Euler's Summation Formula

Tom M. Apostol

1. INTRODUCTION. The integral test for convergence of infinite series compares a finite sum $\sum_{k=1}^{n} f(k)$ and an integral $\int_1^n f(x)\,dx$, where f is positive and strictly decreasing. The difference between a sum and an integral can be represented geometrically, as indicated in Figure 1. In 1736, Euler [3] used a diagram like this to obtain the simplest case of what came to be known as Euler's summation formula, a powerful tool for estimating sums by integrals, and also for evaluating integrals in terms of sums. Later Euler [4] derived a more general version by an analytic method that is very clearly described in [5, pp. 159–161]. Colin Maclaurin [9] discovered the formula independently and used it in his *Treatise of Fluxions*, published in 1742, and some authors refer to the result as the Euler-Maclaurin summation formula. The general formula (24) is widely used in numerical analysis, analytic number theory, and the theory of asymptotic expansions. It contains Bernoulli numbers and periodic Bernoulli functions and is ordinarily discussed in courses in advanced calculus or real and complex analysis. This note shows how the general formula can be discovered by an elementary method, beginning with the diagram in Figure 1. This approach also shows how Bernoulli numbers and Bernoulli functions arise naturally along the way. The author has used this treatment successfully with beginning calculus students acquainted with the integral test.

2. GENERALIZED EULER'S CONSTANT. Throughout this section we assume that f is a positive and strictly decreasing function on $[1, \infty)$. We introduce a sequence $\{d_n\}$ of numbers that represent the sum of the areas of the shaded curvilinear pieces above the interval $[1, n]$ in Figure 1. That is, we define

$$d_n = \sum_{k=1}^{n-1} f(k) - \int_1^n f(x)\,dx, \qquad n = 2, 3, \dots . \tag{1}$$

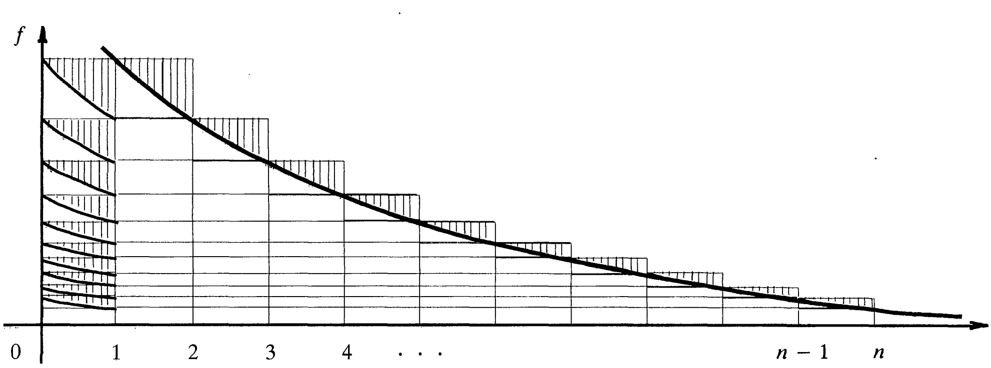

Figure 1. All the shaded regions above $[1, n]$ fit inside a rectangle of area $f(1)$.

It is clear that $d_{n+1} > d_n$ and that all the shaded pieces can be translated to the left to occupy a portion of the rectangle of altitude $f(1)$ above the interval $[0, 1]$, as shown in Figure 1. Because f is decreasing there is no overlapping of the translated shaded pieces. Comparison of areas gives us the inequalities $0 < d_n < d_{n+1} < f(1)$. Therefore $\{d_n\}$ is increasing and bounded above, so it has a finite limit $C(f) = \lim_{n\to\infty} d(n)$. We refer to $C(f)$ as the *generalized Euler's constant* associated with the function f. Geometrically, $C(f)$ represents the sum of the areas of *all* the curvilinear triangular pieces over the interval $[1, \infty)$. These pieces can be translated to fit inside the rectangle of area $f(1)$ shown in Figure 1 (without overlapping), so we have the inequalities $0 < C(f) < f(1)$. Moreover, $C(f) - d_n$ represents the sum of the areas of the triangular pieces over the interval $[n, \infty)$. These pieces can be translated to the left to occupy (without overlapping) a portion of the rectangle of height $f(n)$ above the interval $[n, n+1]$. Comparing areas we find

$$0 < C(f) - d_n < f(n), \qquad n = 2, 3, \dots . \tag{2}$$

From these inequalities we can easily deduce:

Theorem 1. *If f is positive and strictly decreasing on $[1, \infty)$ there is a positive constant $C(f) < f(1)$ and a sequence $\{E_f(n)\}$, with $0 < E_f(n) < f(n)$, such that*

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,dx + C(f) + E_f(n), \quad n = 2, 3, \dots . \tag{3}$$

Note. Eq. (3) tells us that the difference between the sum and the integral is equal to a constant (depending on f) plus a positive quantity $E_f(n)$ smaller than the last term in the sum. Hence, if $f(n)$ tends to 0 as $n \to \infty$, then $E_f(n)$ also tends to 0.

Proof: If we define $E_f(n) = f(n) + d_n - C(f)$, then (3) follows from the definition (1), and the inequality $0 < E_f(n) < f(n)$ follows from (2). ∎

If $f(n) \to 0$ as $n \to \infty$, then (3) implies

$$C(f) = \lim_{n\to\infty}\left(\sum_{k=1}^{n} f(k) - \int_1^n f(x)\,dx\right). \tag{4}$$

Example. When $f(x) = 1/x$, $C(f)$ is the classical *Euler's constant*, often denoted by C (or by γ), and (4) states that $C = \lim_{n\to\infty}(\sum_{k=1}^{n}(1/k) - \log n)$. It is not known (to date) whether Euler's constant is rational or irrational. Its numerical value, correct to 20 decimals, is $C = 0.57721566490153286060$. In this case, Theorem 1 says that

$$\sum_{k=1}^{n} \frac{1}{k} = \log n + C + E(n), \quad \text{where } 0 < E(n) < \frac{1}{n}.$$

3. VARIOUS FORMS OF EULER'S SUMMATION FORMULA. In this section we no longer assume that f is positive or decreasing. At the outset we require only that the integral $\int_1^n f(x)\,dx$ exists for each integer $n \geq 2$. The key insight is to notice that the difference d_n in (1) can be written as

$$d_n = \sum_{k=1}^{n-1} I(k), \tag{5}$$

where

$$I(k) = \int_k^{k+1} \{f(k) - f(x)\}\, dx. \tag{6}$$

When f is positive and decreasing, as in Figure 2, $I(k)$ is the area of the shaded curvilinear triangular piece over the interval $[k, k+1]$. However, (5) and (6) are meaningful for any integrable f.

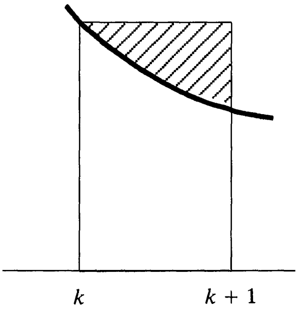

Figure 2. Geometric interpretation of the integral $I(k)$ as the area of the shaded region.

The integrand in (6) has the form $u\,dv$, where $u = f(k) - f(x)$ and $v = x + c$, where c is any constant. If we choose $c = -(k+1)$ and integrate by parts (assuming that f has a continuous derivative), the integrated part vanishes and the integral $I(k)$ reduces to

$$I(k) = \int_k^{k+1} (x - k - 1) f'(x)\, dx.$$

In this integral the dummy symbol x varies from k to $k+1$, so the quantity k in the integrand can be replaced by $[x]$, the greatest integer $\leq x$. Make this replacement and substitute in (6) to find

$$\begin{aligned} d_n &= \sum_{k=1}^{n-1} I(k) = \sum_{k=1}^{n-1} \int_k^{k+1} (x - [x] - 1) f'(x)\, dx \\ &= \int_1^n (x - [x]) f'(x)\, dx - \int_1^n f'(x)\, dx \\ &= \int_1^n (x - [x]) f'(x)\, dx + f(1) - f(n). \end{aligned}$$

Now use the definition of d_n in (1) and rearrange terms to obtain:

Theorem 2. (First-derivative form of Euler's summation formula). *For any function f with a continuous derivative on the interval $[1, n]$ we have*

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\, dx + \int_1^n (x - [x]) f'(x)\, dx + f(1). \tag{7}$$

The last two terms on the right represent the error made when the sum on the left is approximated by the integral $\int_1^n f(x)\, dx$. The formula is useful because f need not be positive or decreasing. In fact, f can be increasing or oscillating. Variants of this formula will be obtained as we attempt to deduce more precise information about the error.

The factor $x - [x]$ is a nonnegative function with period 1. If f' has a fixed sign (as it has when f is monotonic), the integral term in the error has the same sign as f'. To decrease the error it is preferable to multiply $f'(x)$ by a factor that changes

sign so that some cancellation takes place in the integration. To introduce sign changes, we translate the function $x - [x]$ down by $\frac{1}{2}$ and consider the new function $x - [x] - \frac{1}{2}$ whose graph is shown in Figure 3. The integral term in the

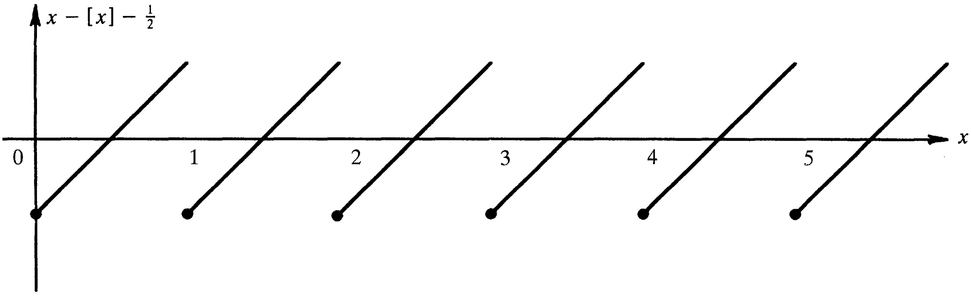

Figure 3. The periodic function $x - [x] - \frac{1}{2}$ changes sign.

error can now be written as

$$\int_1^n (x - [x]) f'(x)\,dx = \int_1^n \left(x - [x] - \frac{1}{2}\right) f'(x)\,dx + \frac{1}{2}\int_1^n f'(x)\,dx.$$

The last term is equal to $\frac{1}{2}\{f(n) - f(1)\}$. Using this in (7) we obtain the following variant of the first-derivative form of Euler's summation formula:

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,dx + \int_1^n \left(x - [x] - \frac{1}{2}\right) f'(x)\,dx + \frac{1}{2}\{f(n) + f(1)\}. \quad (8)$$

Further variations will be obtained by repeated integration by parts in the second integral on the right of (8).

The factor $x - [x] - \frac{1}{2}$ has the value $-\frac{1}{2}$ when x is an integer. We modify this factor slightly to make it vanish at the integers, a property that is desirable when we integrate by parts. To do this we introduce $P_1(x)$, the *first Bernoulli function*:

$$P_1(x) = \begin{cases} x - [x] - \dfrac{1}{2} & \text{if } x \neq \text{integer} \\ 0 & \text{if } x = \text{integer}. \end{cases} \quad (9)$$

The error integral does not change if the factor $x - [x] - \frac{1}{2}$ is replaced by $P_1(x)$ because the two factors differ only at the integers. Therefore (8) can be written as

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,dx + \int_1^n P_1(x) f'(x)\,dx + \frac{1}{2}\{f(n) + f(1)\}. \quad (10)$$

Note the contrast between (10) and (3), which explicitly displays the generalized Euler's constant $C(f)$. To make (10) resemble (3) more closely, we assume that the improper integral $\int_1^\infty P_1(x) f'(x)\,dx$ converges. Then we can write

$$\int_1^n P_1(x) f'(x)\,dx = \int_1^\infty P_1(x) f'(x)\,dx - \int_n^\infty P_1(x) f'(x)\,dx,$$

and (10) takes the form

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,dx + C(f) + E_f(n), \quad (11)$$

where

$$C(f) = \frac{1}{2}f(1) + \int_1^{\infty} P_1(x) f'(x)\, dx \tag{12}$$

and

$$E_f(n) = \frac{1}{2}f(n) - \int_n^{\infty} P_1(x) f'(x)\, dx.$$

Eq. (11) has exactly the same form as (3), but (11) is more general because f is not required to be positive or monotonic. The only restrictions on f are continuity of f' and convergence of the improper integral

$$\int_1^{\infty} P_1(x) f'(x)\, dx. \tag{13}$$

The improper integral in (13) converges if, and only if,

$$\lim_{n\to\infty} \int_n^{\infty} P_1(x) f'(x)\, dx = 0. \tag{14}$$

A sufficient condition for convergence is that $\int_1^{\infty} |f'(x)|\, dx$ converges, or equivalently, that

$$\lim_{n\to\infty} \int_n^{\infty} |f'(x)|\, dx = 0. \tag{15}$$

To see this, note that the Bernoulli function $P_1(x)$ is bounded; in fact, Figure 3 shows that $|P_1(x)| \le \frac{1}{2}$ for all x, so (14) follows from (15).

Example. When $f(x) = 1/x$ we have $f'(x) = -1/x^2$ and

$$\int_n^{\infty} |f'(x)|\, dx = \int_n^{\infty} \frac{1}{x^2}\, dx = \frac{1}{n}.$$

Therefore (15) is satisfied and (12) expresses Euler's constant as an integral:

$$C = \frac{1}{2} - \int_1^{\infty} \frac{P_1(x)}{x^2}\, dx.$$

4. FURTHER ANALYSIS OF THE ERROR TERM. Alternate forms of both the error term and the formula for the generalized Euler's constant can be obtained by repeated integration by parts. First we introduce a new function $P_2(x)$ whose derivative is $2P_1(x)$ at all noninteger values of x. The factor 2 is used so that $P_2(x)$ is the second Bernoulli periodic function that appears in Euler's summation formula. Therefore we require that

$$P_2(x) = 2\int_0^x P_1(t)\, dt + c, \tag{16}$$

where c is a constant to be specified later. The function P_2 is quadratic on the interval $[0, 1]$. In fact, $P_2(x) = x^2 - x + c$ if $0 \le x \le 1$. Its graph is a parabolic arc joining the points $(0, c)$ and $(1, c)$. Outside this interval the graph (shown in Figure 4) consists of horizontal translations of this parabolic arc because P_2 has period 1. To see this, we use the fact that P_1 has period 1 and that $\int_0^1 P_1(t)\, dt = 0$, which implies that $\int_a^b P_1(t)\, dt = 0$ for any interval $[a, b]$ of length 1. Therefore

$$P_2(x+1) - P_2(x) = 2\int_x^{x+1} P_1(t)\, dt = 0.$$

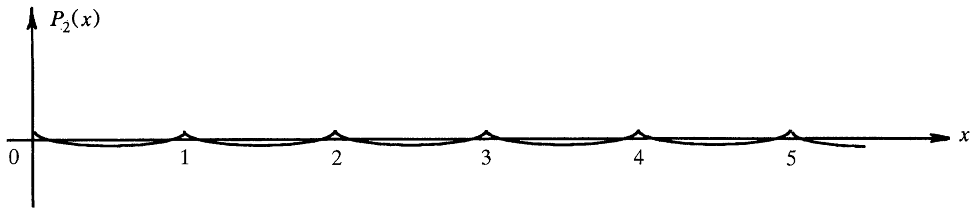

Figure 4. Graph of $P_2(x) = 2\int_0^x P_1(t)\,dt + c$.

Because of periodicity, P_2 has the constant value $c = P_2(0)$ at the integers. Integration by parts shows that the integral in (10) is

$$\int_1^n P_1(x)f'(x)\,dx = \frac{1}{2}P_2(0)\{f'(n) - f'(1)\} - \frac{1}{2}\int_1^n P_2(x)f''(x)\,dx,$$

provided that f'' is continuous. Repeated integration by parts leads to the general form of Euler's summation formula, which involves higher order derivatives of f and higher order periodic Bernoulli functions that represent polynomials on the unit interval $[0, 1]$. To see exactly how the Bernoulli functions evolve in the process we follow the method of the foregoing section and integrate the periodic function $3P_2(t)$ from 0 to x to obtain another periodic function $P_3(x)$ whose derivative is $3P_2(x)$. To guarantee that the integrated function $P_3(x)$ is periodic with period 1 we need $\int_0^1 P_2(t)\,dt = 0$. This property governs the choice of the constant c in (16). The integral of the quadratic polynomial $x^2 - x + c$ from 0 to 1 is equal to $c - \frac{1}{6}$, so we choose $c = \frac{1}{6}$ and take

$$P_2(x) = 2\int_0^x P_1(t)\,dt + \frac{1}{6}.$$

Euler's summation formula can now be restated as follows:

Theorem 3. (Second-derivative form of Euler's summation formula). *For any function f with a continuous second derivative on the interval $[1, n]$ we have*

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,dx - \frac{1}{2}\int_1^n P_2(x)f''(x)\,dx + \frac{1}{2}P_2(0)\{f'(n) - f'(1)\} + \frac{1}{2}\{f(n) + f(1)\}. \tag{17}$$

Moreover, if the improper integral $\int_1^\infty |f''(x)|\,dx$ converges then we also have

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,dx + C(f) + E_f(n),$$

where

$$C(f) = \frac{1}{2}f(1) - \frac{1}{2}P_2(0)f'(1) - \frac{1}{2}\int_1^\infty P_2(x)f''(x)\,dx, \tag{18}$$

and

$$E_f(n) = \frac{1}{2}f(n) + \frac{1}{2}P_2(0)f'(n) + \frac{1}{2}\int_n^\infty P_2(x)f''(x)\,dx. \tag{19}$$

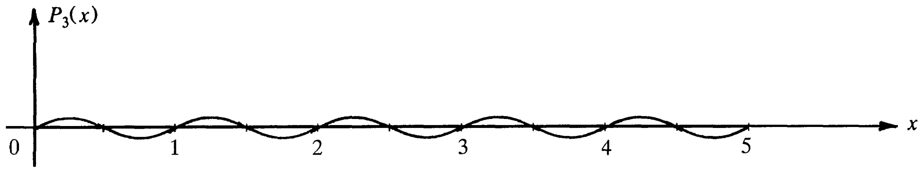

Figure 5. Graph of the periodic Bernoulli function $P_3(x) = 3\int_0^x P_2(t)\,dt$.

To improve the error estimate we integrate $P_2(t)$ from 0 to x and define the Bernoulli function $P_3(x) = 3\int_0^x P_2(t)\,dt$ so that $P_3'(x) = 3P_2(x)$. There is no need to add a constant in this case because, on the unit interval $[0, 1]$, $P_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$, and $\int_0^1 P_3(t)\,dt = 0$. The function P_3 has period 1 because P_2 has period 1 and $\int_0^1 P_2(t)\,dt = 0$. The graph of P_3 is a bounded piecewise cubic curve, shown in Figure 5. Note that $P_3(x)$ vanishes at the integers. Integration by parts over $[1, n]$ gives us

$$\int_1^n P_2(x)f''(x)\,dx = -\frac{1}{3}\int_1^n P_3(x)f^{(3)}(x)\,dx,$$

provided $f^{(3)}$ is continuous. This equation, together with Theorem 3, gives a third-derivative form of Euler's summation formula in which the second integral on the right of (17) is replaced by $\frac{1}{3!}\int_1^n P_3(x)f^{(3)}(x)\,dx$. The corresponding changes in (18) and (19) are replacement of the integrals by $\frac{1}{3!}\int_1^\infty P_3(x)f^{(3)}(x)\,dx$ and $\frac{-1}{3!}\int_n^\infty P_3(x)f^{(3)}(x)\,dx$, respectively.

5. BERNOULLI NUMBERS AND THE GENERAL FORM OF EULER'S SUMMATION FORMULA. The strategy for obtaining a general version of Euler's summation formula is now evident. Starting with the Bernoulli periodic function $P_1(x)$ in (9) we introduce, in succession, periodic functions $P_2(x), P_3(x), \ldots,$ with period 1, and a sequence of constants B_k such that

$$P_k(x) = k\int_0^x P_{k-1}(t)\,dt + B_k \quad \text{for } k \geq 2, \tag{20}$$

where each B_k is chosen so that

$$\int_0^1 P_k(t)\,dt = 0. \tag{21}$$

Periodicity implies that $P_k(0) = P_k(1)$, and (21) shows that each of these values is B_k. As already noted, on the closed interval $[0, 1]$ each function $P_k(x)$ is a polynomial of degree k when $k = 2$ or 3. [The case $k = 1$ is special; $P_1(x)$ is a linear polynomial $x - \frac{1}{2}$ only on the open interval $(0, 1)$ and is discontinuous at the endpoints.] It is clear (and easily proved by induction) that on the closed interval $[0, 1]$ the function defined by (20) is a polynomial of degree k if $k \geq 2$. We denote this polynomial by $B_k(x)$, the usual notation for *Bernoulli polynomials*. The first few are

$$B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6}, \quad B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x,$$
$$B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}, \quad B_5(x) = x^5 - \frac{1}{2}x^4 + \frac{1}{3}x^3 - \frac{1}{30}x,$$
$$B_6(x) = x^6 - \frac{3}{5}x^5 + \frac{1}{2}x^4 - \frac{1}{10}x^2 + \frac{1}{42}.$$

The *Bernoulli periodic functions* are periodic extensions of these polynomials given by $P_k(x) = B_k(x - [x])$. The constants $B_k = P_k(0) = P_k(1)$ are called *Bernoulli numbers*. The first few are

$$B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30},$$

$$B_5 = 0, \quad B_6 = \frac{1}{42}, \quad B_7 = 0, \quad B_8 = -\frac{1}{30}.$$

Next we show that our definitions of Bernoulli numbers and polynomials are consistent with the usual definitions, provided we take $B_0(x) = 1$ and $B_0 = 1$. Our definition in (20) shows that the successive derivatives of these polynomials are

$$B_k'(x) = kB_{k-1}(x), B_k''(x) = k(k-1)B_{k-2}(x), \ldots, B_k^{(r)}(x) = r!\binom{k}{r}B_{k-r}(x),$$

and hence

$$B_k^{(r)}(0) = r!\binom{k}{r}B_{k-r}(0) = r!\binom{k}{r}B_{k-r}. \tag{22}$$

On the other hand, the Taylor expansion of any polynomial $B_k(x)$ of degree k is given by $B_k(x) = \sum_{r=0}^{k} B_k^{(r)}(0)x^r/r!$, so (22) implies

$$B_k(x) = \sum_{r=0}^{k} \binom{k}{r} B_{k-r} x^r. \tag{23}$$

Taking $x = 1$ in (23) and noting that $B_k(1) = P_k(1) = B_k$ for $k \geq 2$, we find that (23) becomes

$$B_k = \sum_{r=0}^{k} \binom{k}{r} B_{k-r} \quad \text{for } k \geq 2.$$

This is the usual recursion formula for defining Bernoulli numbers (starting with $B_0 = 1$), and (23) is one of the standard ways of defining Bernoulli polynomials in terms of Bernoulli numbers. Consequently, the numbers and polynomials that appear in our treatment are the usual Bernoulli numbers and Bernoulli polynomials that appear in the literature; see [**1**, p. 265], [**2**, p. 251], or [**5**, pp. 160–163].

It is well known that the Bernoulli numbers B_k with odd index $k \geq 3$ are zero, so only Bernoulli numbers with even index appear in the general form of Euler's summation formula. It is also known [**8**, p. 533] that on the interval $[0, 1]$ the Bernoulli polynomials satisfy the following inequalities for $k \geq 1$:

$$|B_{2k}(x)| \leq |B_{2k}| \quad \text{and} \quad |B_{2k+1}(x)| \leq (2k+1)|B_{2k}|.$$

The method we have outlined leads to the following odd-order derivative version of Euler's summation formula. A proof is easily given by induction on the order $2m + 1$.

Theorem 4. (General form of Euler's summation formula). *For any function f with a continuous derivative of order $2m + 1$ on the interval $[1, n]$ we have*

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,dx + \frac{1}{(2m+1)!}\int_1^n P_{2m+1}(x) f^{(2m+1)}(x)\,dx$$
$$+ \sum_{r=1}^{m} \frac{B_{2r}}{(2r)!}\{f^{(2r-1)}(n) - f^{(2r-1)}(1)\} + \frac{1}{2}\{f(1) + f(n)\}. \tag{24}$$

Moreover, if the improper integral $\int_1^\infty |f^{(2m+1)}(x)|\,dx$ *converges then we also have*

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,dx + C(f) + E_f(n), \tag{25}$$

where

$$C(f) = \frac{1}{2}f(1) - \sum_{r=1}^{m} \frac{B_{2r}}{(2r)!} f^{(2r-1)}(1) + \frac{1}{(2m+1)!}\int_1^\infty P_{2m+1}(x) f^{(2m+1)}(x)\,dx, \tag{26}$$

and

$$E_f(n) = \frac{1}{2}f(n) + \sum_{r=1}^{m} \frac{B_{2r}}{(2r)!} f^{(2r-1)}(n) - \frac{1}{(2m+1)!}\int_n^\infty P_{2m+1}(x) f^{(2m+1)}(x)\,dx. \tag{27}$$

Example. When $f(x) = 1/x$ we have $f^{(2m+1)}(x) = -(2m+1)!/x^{2m+2}$, and (26) gives the following expression for the classical Euler's constant:

$$C = \frac{1}{2} + \frac{B_2}{2} + \frac{B_4}{4} + \cdots + \frac{B_{2m}}{2m} - \int_1^\infty \frac{P_{2m+1}(x)}{x^{2m+2}}\,dx. \tag{28}$$

The corresponding error term (27) becomes

$$E_f(n) = \frac{1}{2n} - \frac{B_2}{2n^2} - \frac{B_4}{4n^4} - \cdots - \frac{B_{2m}}{2mn^{2m}} + \int_n^\infty \frac{P_{2m+1}(x)}{x^{2m+2}}\,dx. \tag{29}$$

One is tempted to let $m \to \infty$ in (28) and obtain an infinite series for Euler's constant. However, the integral in (28) does not tend to 0 as $m \to \infty$ and, in fact, it can be shown that the infinite series $\Sigma B_{2k}/(2k)$ diverges rapidly [see **6**, p. 529], so (28) is not very useful for calculating C. Nevertheless, as we show in the next section, (25) and (27) can be used to calculate C very accurately.

6. CALCULATION OF EULER'S CONSTANT. We use Euler's summation formula to calculate the first 7 digits in Euler's constant. Take $f(x) = 1/x$ in (25) and rewrite it as

$$C = \sum_{k=1}^{n} \frac{1}{k} - \log n - E_f(n), \tag{30}$$

where $E_f(n)$ is given by (29). Taking $m = 3$ in (29) we find

$$\begin{aligned} E_f(n) &= \frac{1}{2n} - \frac{B_2}{2n^2} - \frac{B_4}{4n^4} - \frac{B_6}{6n^6} + \int_n^\infty \frac{P_7(x)}{x^8}\,dx \\ &= \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \frac{1}{252n^6} + \int_n^\infty \frac{P_7(x)}{x^8}\,dx. \end{aligned}$$

Using the inequality $|P_7(x)| \le 7|B_6| = \frac{1}{6}$, we get

$$\left|\int_n^\infty \frac{P_7(x)}{x^8}\,dx\right| \le \frac{1}{6}\int_n^\infty \frac{1}{x^8}\,dx = \frac{1}{42n^7},$$

and (30) can be written in the form

$$C = \sum_{k=1}^{n} \frac{1}{k} - \log n - \frac{1}{2n} + \frac{1}{12n^2} - \frac{1}{120n^4} + \frac{1}{252n^6} + E(n), \tag{31}$$

where $0 < |E(n)| \leq 1/42n^7$. Using a hand calculator that displays 12 digits we find $\sum_{k=1}^{10} k^{-1} = 2.92896825381$ and $\log 10 = 2.30258509299$. If $n = 10$ the sum of the error term $E(n)$ plus the term with $252n^6$ in the denominator in (31) is too small to influence the seventh digit. Neglecting these terms and retaining 8 digits in the calculation we find

$$\begin{aligned} C &\doteq 2.92896825 - 2.30258509 - \frac{1}{20} + \frac{1}{1200} - \frac{1}{1200000} \\ &= 0.62638316 - 0.05000000 + 0.00083333 - 0.00000083 \\ &= 0.57721566 \end{aligned}$$

This calculation, using $m = 3$ and $n = 10$ in (29) and (30), which guarantees 7 decimal places, actually gives the first 8 correct digits of C. Knuth [**7**] used (29) and (30) with $m = 250$ and $n = 10{,}000$ to calculate the value of C to 1,271 decimal places.

This note outlines only one application of Euler's summation formula. Others can be found in Knopp's treatise [**6**]. One of them uses the increasing function $f(x) = \log x$ to derive Stirling's asymptotic formula for the logarithm of $n!$. Euler's summation formula and its relation to Bernoulli numbers and polynomials provides a treasure trove of interesting enrichment material suitable for elementary calculus courses.

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Marriage, Magic, and Solitaire

David B. Leep and Gerry Myerson

1. SOLITAIRE. Here's a solitaire game you can always win.

Deal out a deck of cards, face up, into a 4×13 array. The object of the game is to select 13 cards, one from each column, in such a way as to get one card of each denomination.

It turns out that it is always possible to make such a selection. The proof is a simple application of Hall's Marriage Theorem, as we show in Example 1 in the next section. In Sections 3 and 4, we identify winning the solitaire game with decomposing a semi-magic square into a linear combination, with positive integer coefficients, of permutation matrices. The remainder of the paper discusses the number of permutation matrices needed to express a given semi-magic square.

2. MARRIAGE. Suppose there are sets $A_1, A_2, \dots, A_n$, and you wish to know whether there exist distinct objects $x_1, x_2, \dots, x_n$, such that x_1 is in A_1, x_2 is in $A_2, \dots$, and x_n is in A_n—we'll call this a *transversal.* If any A_j is empty, then it's clear that x_j does not exist; a simple necessary condition for the existence of a transversal is that $\#A_j \geq 1$ for all j—we write $\#S$ for the cardinality of the set S.

If among the sets $A_1, \dots, A_n$ there are two whose union has only one element, then there can be no transversal. More generally, a necessary condition for the existence of a transversal is that $\# \bigcup_{j \in J} A_j \geq \#J$ for every index set $J \subset \{1, \dots, n\}$.

Hall's Marriage Theorem states that this simple necessary condition is also sufficient:

Theorem 1. *There exist distinct $x_1, \dots, x_n$ such that $x_j \in A_j$ for all j if and only if $\# \bigcup_{j \in J} A_j \geq \#J$ for all $J \subset \{1, \dots, n\}$.*

Many proofs are known, and the reader with access to combinatorics and/or graph theory textbooks will have little difficulty finding one, so we do not present one here. The compilation [2] contains Hall's original proof, and the spiffy proof of Halmos and Vaughan. The interpretation wherein the "objects" are men and A_j is the set of suitable marriage partners for the jth woman is the origin of the name, "Marriage Theorem."

The application to the solitaire game is as follows.

Example 1. Let the objects be the 13 denominations, and let A_j be the set of all denominations of cards in the jth column. For example, if column 7 has an ace, a deuce, and two jacks, then $A_7 = \{\text{ace, deuce, jack}\}$. Any collection of k columns, $1 \leq k \leq 13$, contains $4k$ cards, hence contains cards of at least k different denominations (since there are only 4 cards of each denomination). But this is precisely the condition for Hall's Theorem to apply, and it tells us we can choose a different denomination from each column.

3. MAGIC. That could be the end of the discussion, but instead we approach the problem from a different point of view, in order to introduce the topic we really want to talk about: semi-magic squares. Much of what we have to say applies, mutatis mutandis, to doubly-stochastic matrices, so there should be something here to appeal to a variety of mathematical tastes.

Having dealt out the cards, construct a 13×13 matrix A, as follows. Each column in A corresponds to a column of cards, and each row to a denomination. The value of a_{ij} (the usual notation for the entry in row i, column j of A, although we also write $A(i, j)$) is then taken to be the number of cards of denomination i in column j. In Example 1, we would have $a_{\text{ace}, 7} = 1$, $a_{\text{jack}, 7} = 2$, and $a_{\text{queen}, 7} = 0$.

The matrix so constructed enjoys the following properties;

1. its entries are non-negative integers,
2. the entries in each row add up to 4 (because there are exactly 4 cards of each denomination), and
3. the entries in each column also add up to 4 (because there are exactly 4 cards in each column of cards).

Thus, the matrix is a *semi-magic square*; a square array of non-negative integers having constant line-sums. "Line-sums" means both row and column sums. The common value of the line-sums is called the *magic constant* of the semi-magic square, and is denoted by m. In a *magic square*, the entries along each diagonal also add up to m, but we do not invoke this condition in the sequel.

Hall's Theorem has the following consequence:

Theorem 2. *A non-zero semi-magic square has a transversal all of whose elements are non-zero.*

In this context, "transversal" means a set of entries meeting each line exactly once (that is, one entry from each column, each from a different row). For, let the columns correspond to sets, and the rows to objects, and let a_{ij} non-zero mean that object i is in set j. In any k columns, the non-zero entries add up to km. Restricting our attention to those k columns, if fewer than k rows meet those columns in non-zero entries, then at least one row meets those columns in entries that add up to more than m; but this is impossible, since the entries in each row add up to exactly m. Thus, Hall's Theorem applies, and there is a choice of a different object from each set; a non-zero entry from each column, each from a different row.

In the 13×13 semi-magic square constructed in Example 1 from an array of cards, a transversal corresponds to a selection of one card from each column, each of a different denomination. Thus we have a second way to use Hall's Theorem to prove that we can always win this game of solitaire.

4. PERMUTATIONS. Perhaps the simplest non-zero semi-magic squares are those with all line-sums 1, the *permutation matrices*. A permutation matrix is a matrix of zeros and ones, the ones forming a transversal. The name arises from the association of each such matrix A to a permutation σ via $a_{ij} = 1$ if and only if $\sigma(i) = j$. This association is a group isomorphism from the multiplicative group of $n \times n$ permutation matrices to the group of all permutations of $\{1, \ldots, n\}$.

We can reformulate Theorem 2: if A is a non-zero semi-magic square, then there is a permutation matrix P such that $A - P$ has non-negative entries. But then $A - P$ is itself clearly a semi-magic square, whence, by induction, we deduce

Theorem 3. *Every semi-magic square can be expressed as a sum of permutation matrices.*

Theorems 2 and 3 are due to Kőnig [**3**]. As an illustration of Theorem 3, we note that

$$\begin{pmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{pmatrix} = 7\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + 4\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$
$$+ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

A *doubly-stochastic* matrix is a matrix with non-negative real entries and all line-sums equal to one. Dividing any non-zero semi-magic square by its magic constant yields a doubly-stochastic matrix. Birkhoff [**1**] proved that every doubly-stochastic matrix is a convex combination of permutation matrices; see also [**6**, Theorem 5.4 of Chapter 5].

An expression of a semi-magic square as a sum of permutation matrices is, in general, not unique. We may ask for an expression that uses as few distinct permutation matrices as possible. The rest of this paper is an attempt to come to grips with this and related questions.

5. THE BASIS. The concepts of permutation matrix and semi-magic square generalize readily to square matrices with entries from any ring R with unit. Let the unit element of R be 1. Then a permutation matrix over R is, as before, a matrix of zeros and ones, the ones forming a transversal. A constant line-sum matrix over R is a square array of elements of R having all line-sums equal. We reserve the term "semi-magic square" for a constant line-sum matrix over the integers with non-negative entries. Any linear combination of permutation matrices with coefficients in R is a constant line-sum matrix over R. We have seen that any semi-magic square with non-negative integer entries is an integer-linear combination of permutation matrices, and we now show that this, too, generalizes to constant line-sum matrices over R. The case $n = 1$ is trivial, and a 2×2 constant line-sum matrix must look like

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus, we may assume $n \geq 3$. Let $\mathscr{B}_n$ be the set of all permutation matrices corresponding to those transpositions and 3-cycles that move 1, together with the identity matrix. That is, $\mathscr{B}_n$ contains the permutations of the form $(1j)$, $2 \leq j \leq n$, and those of the form $(1jk)$, $2 \leq j \leq n$, $2 \leq k \leq n$, $j \neq k$, and the identity. Then $\mathscr{B}_n$ is a linearly independent set, over any ring whatsoever. For if $\sum_{P_\sigma \in \mathscr{B}_n} a_\sigma P_\sigma = 0$, then $a_{(1jk)}$ must be zero, since $P_{(1jk)}$ is the only matrix in $\mathscr{B}_n$ with a non-zero entry in row j, column k. And if all the $a_{(1jk)}$ are zero, then $a_{(1j)}$ must be zero, since each $P_{(1j)}$ has a one in the first row that none of the others has. Finally, $a_{()}$ must be zero.

Now let A be any $n \times n$ constant line-sum matrix over R. Let $B = A - \sum_{j,k} a_{jk} P_{(1jk)}$, taking the sum over all j and k distinct from each other and one; then $b_{jk} = 0$ for all these j and k. Let $C = B - \sum_{j=1}^{n} b_{1j} P_{(1j)}$. Then C has all line-sums zero, since its first row is entirely zeros. Each column of C, other than

the first, has $n-1$ zeros, hence, n zeros; then, looking across the rows, we see that all the entries in the first column must be zero as well. Thus, $A = \Sigma a_{jk} P_{(1jk)} + \Sigma b_{1j} P_{(1j)}$ expresses A as a linear combination of permutation matrices (with coefficients in the ring generated by the entries of A).

Summing up, we have proved:

Theorem 4. *For any ring R with unit, the set of all R-linear combinations of elements of $\mathscr{B}_n$ is the set of all $n \times n$ constant line-sum matrices with entries in R.*

A closer look at the proof leads to our next result.

Theorem 5. *Each $n \times n$ permutation matrix can be written as a ± 1-combination of at most $2n-1$ elements of $\mathscr{B}_n$ (meaning, a linear combination in which each coefficient is 1 or -1).*

Proof: Let A in the proof of Theorem 4 be a permutation matrix. For each j, $2 \le j \le n$, there is at most one k, $k \ne 1$, $k \ne j$, such that $a_{jk} = 1$. Thus, at most $n-1$ of the coefficients $a_{(1jk)}$ are one (the rest being zero), and no two of these have the same value of j. So, the first row of $B = A - \Sigma a_{(1jk)} P_{(1jk)}$ takes all n of its entries from $\{-1, 0, 1\}$, and $b_{(1j)}$ is in $\{-1, 0, 1\}$ for all j. ■

That Theorems 4 and 5 proclaim a special property of $\mathscr{B}_n$ can be seen from the following equation, valid for any $n \ge 4$:

$$2I = (12) + (23) + (34) + (41) - (1234) - (4321), \tag{1}$$

where we have adopted the notational convenience of replacing a permutation matrix with the permutation it represents. It is easy to check that the six matrices on the right are linearly independent over any ring R that does not have a non-zero element x satisfying $x + x = 0$; but the identity matrix cannot be expressed as a ± 1-combination of any linearly independent set that includes these six matrices, and it cannot be written as an R-linear combination at all, if R has no element x satisfying $x + x = 1$ (for example, if R is the integers).

This example suggests a question, for which we do not know the answer: given n, for which integers m does there exist an $n \times n$ semi-magic square A, a linearly independent set of permutation matrices $\{P_1, \ldots, P_r\}$, and non-negative integers $c_1, \ldots, c_r$ with $\gcd(c_1, \ldots, c_r) = 1$, such that $mA = \Sigma_j c_j P_j$? Equation (1) shows that we may take $m = 2$ for every $n \ge 4$; indeed, from

$$\begin{aligned}(m-2)I = (12) + (23) + \cdots + (m-1\ m) + (m\ 1)\\ -(12\ldots m) - (m\ m-1\ldots 1)\end{aligned}$$

it is easy to verify that for any n we can take any m not exceeding $n-2$.

6. HOW MANY? (BIG FIELDS). It is easy to see that the set of all $n \times n$ constant line-sum matrices over a field F forms a vector space over F. What is the dimension of this space?

Theorem 6. *The dimension of the vector space of all $n \times n$ constant line-sum matrices over a field F is $n^2 - 2n + 2$.*

This can be seen in several different ways.

1) Assign arbitrary values to a_{ij}, $1 \le i \le n-1$, $1 \le j \le n-1$, and also to a_{1n}, making $(n-1)^2 + 1$ arbitrary choices in all. There is a unique choice of each a_{in},

$2 \le i \le n - 1$, and each a_{nj}, $1 \le j \le n - 1$, that makes the corresponding row or column sum equal to the sum of the entries in the first row, and then a unique choice of a_{nn} to complete the constant line-sum matrix.

2) To be a constant line-sum matrix is to satisfy $2n - 1$ equations of the form, "the entries in row 1 add up to the same number as the entries in a different line." There is one dependence relation among these equations, since the sum of all the row sums equals the sum of all the column sums, so the vector space has codimension $2n - 2$ in the vector space of all $n \times n$ matrices, which means that the dimension is $n^2 - (2n - 2)$.

3) The basis $\mathscr{B}_n$ has $(n - 1)(n - 2)$ elements of the form $(1jk)$, $n - 1$ of the form $(1j)$, and the identity, making $n^2 - 2n + 2$ in all.

It follows from Theorems 4 and 6 that, over a field, any $n \times n$ constant line-sum matrix can be expressed as a linear combination of $n^2 - 2n + 2$ or fewer permutation matrices. It also follows that, over an infinite field (or, indeed, a sufficiently large finite field), there exist constant line-sum matrices that cannot be expressed as a linear combination of fewer than $n^2 - 2n + 2$ permutation matrices. This is based on the observation that no vector space over an infinite field is the union of finitely many proper subspaces, which is a corollary to a technical lemma that we have relegated to the appendix.

7. HOW MANY? (NON-NEGATIVE INTEGERS) (THEORY). Life is somewhat different over a (small) finite field, but we postpone discussion of that situation until we have considered the integers. Results about linear combinations with positive integer coefficients do not follow trivially from results about fields, but they do follow:

Theorem 7. *Each $n \times n$ semi-magic square can be expressed as a linear combination, with positive integer coefficients, of $n^2 - 2n + 2$ or fewer permutation matrices.*

Proof: We follow the argument by which Marcus and Ree [**5**] proved that every doubly-stochastic matrix is a convex combination of $n^2 - 2n + 2$ or fewer permutation matrices. Let A be a non-zero $n \times n$ semi-magic square (if $A = 0$, there is nothing to prove). By Theorem 2 we know there is a permutation matrix P_1 such that $A - P_1$ has non-negative integer entries. Choose m_1 as large as possible, subject to $A_1 = A - m_1 P_1$ having non-negative entries. Note that P_1 has a one in some spot where A_1 has a zero and that the magic constant of A_1 is strictly less than that of A. Now apply the same procedure to A_1, and iterate to termination. Termination must occur, since the magic constants form a strictly decreasing sequence of non-negative integers. When the procedure terminates, we have $A = m_1 P_1 + \cdots + m_r P_r$ for some r. But the matrices $P_1, \dots, P_r$ are linearly independent (over, say, the rationals), since each has a one in a spot where its successors all have zero. So, r is no greater than the dimension of the space spanned by all the $n \times n$ permutation matrices, and we know from Section 6 that this dimension is $n^2 - 2n + 2$. ∎

We would like to know whether there is an "integer proof" of Theorem 7, that is, a proof that does not rely on embedding the integers into a field and using dimension, a vector space concept.

Theorem 8. *For every n there exist $n \times n$ semi-magic squares that cannot be expressed as a linear combination, with non-negative integer coefficients, of $n^2 - 2n + 1$ permutation matrices.*

We give three proofs.

First proof: Let A be an $n \times n$ constant line-sum matrix with non-negative rational entries, and assume that A is not a linear combination with rational coefficients of $n^2 - 2n + 1$ permutation matrices. Such matrices exist by a corollary to the technical lemma in the appendix. Let m be a common multiple of the denominators of the entries of A. Then mA is a semi-magic square, and is not expressible as a linear combination with rational coefficients (nor, *a fortiori*, with non-negative integer coefficients) of $n^2 - 2n + 1$ permutation matrices. For, if there were such an expression for mA, then dividing through by m would give an expression for A as a rational linear combination of $n^2 - 2n + 1$ permutation matrices. ■

Second proof: We count the number of $n \times n$ semi-magic squares with magic constant N, and the number of linear combinations of $n^2 - 2n + 1$ permutation matrices with positive integer coefficients adding up to N, and we see that, if N is large enough, there are too many of the former to be accounted for by the latter.

Given integers a_{ij} with $N(n-2)/(n-1)^2 \le a_{ij} \le N/(n-1)$ for $1 \le i \le n-1$ and $1 \le j \le n-1$, there exist non-negative integers a_{in}, $1 \le i \le n$, and a_{nj}, $1 \le j \le n$, such that A is a semi-magic square with magic constant N. Thus, the number of squares with magic constant N is at least $c_1 N^{(n-1)^2}$. Here and in the following discussion $c_1, c_2, \ldots$ depend on n but not on N, and the exact nature of the dependence is irrelevant.

To count the number of non-negative integer linear combinations of $n^2 - 2n + 1$ permutation matrices, with all line-sums equal to N, we note first that there are $\binom{n!}{(n-1)^2} = c_2$ ways of choosing the permutation matrices. Having chosen them, we have only to count the number of expressions $\sum_{j=1}^{n^2-2n+1} a_j P_j$ subject to the conditions $\sum a_j = N$ and $a_j \ge 0$ for all j. But the number of ways to meet the conditions is $\binom{N + (n-1)^2 - 1}{(n-1)^2 - 1}$, which is a polynomial in N of degree $(n-1)^2 - 1$ and is thus bounded above by $c_3 N^{(n-1)^2 - 1}$ for some c_3. So, the total number of semi-magic squares of magic constant N representable as non-negative integer linear combinations of $n^2 - 2n + 1$ permutation matrices is at most $c_4 N^{(n-1)^2-1}$, where $c_4 = c_2 c_3$. If N is large enough, $c_1 N^{(n-1)^2} > c_4 N^{(n-1)^2-1}$, so there must be semi-magic squares that cannot be expressed as a non-negative integer linear combination of $n^2 - 2n + 1$ permutation matrices. ■

We could use this second proof to estimate the value of N needed, but we have thrown too much away for the estimate to be any good. Our third proof actually constructs the object whose existence is established by the first two proofs.

Third proof: Let $P_1, \ldots, P_d$, $d = n^2 - 2n + 2$, be the special basis $\mathscr{B}_n$ discussed in Section 5, ordered in such a way that all the 3-cycles come first, then the transpositions, finally, the identity. Let $A = \sum_{j=1}^{d} c_j P_j$, where c_j is any sequence of positive integers growing fast enough to satisfy $c_j > \sum_{k=1}^{j-1}(j-k)c_k$ for all j (the sequence $1, 2, 5, 13, 34, \ldots$ of alternate Fibonacci numbers will do, barely). We claim that A cannot be expressed as a positive integer linear combination of fewer than $n^2 - 2n + 2$ permutation matrices.

Recall that each P_j has a "special spot" where it has a one and where each P_k, $k > j$, has a zero. Given any j, and any matrix B, we write $B(j)$ for the entry of B in the special spot of P_j.

Let $A = \sum_{j=1}^{r} a_j Q_j$ for some positive integers a_j and some permutation matrices Q_j. Since $A(1) = c_1 \geq 1$, we must have $Q_j(1) = 1$ for some j. Re-ordering, if necessary, we may assume $Q_1(1) = 1$. It follows that $a_1 = c_1$. Let $A_1 = A - a_1 Q_1$.

Now suppose that for $1 \leq j \leq k-1$ we have $Q_j(j) = 1$, $1 \leq a_j \leq \sum_{t=1}^{j} c_t$, and $A_j = A_{j-1} - a_j Q_j = A - a_1 Q_1 - \cdots - a_j Q_j$. Note that $c_k \leq A(k) \leq \sum_{j=1}^{k} c_j$. It follows that

$$1 \leq c_k - \sum_{j=1}^{k-1} (k-j) c_j = c_k - \sum_{j=1}^{k-1} \sum_{t=1}^{j} c_t \leq c_k - \sum_{j=1}^{k-1} a_j \leq A_{k-1}(k) \leq \sum_{j=1}^{k} c_j.$$

Since $A_{k-1}(k) \geq 1$, we must have $Q_j(k) = 1$ for some $j \geq k$. Re-ordering, if necessary, we may assume $Q_k(k) = 1$. Then $1 \leq a_k \leq \sum_{t=1}^{k} c_t$.

By induction, we see that $Q_j(j) = 1$ for $1 \leq j \leq n^2 - 2n + 2$, and $r = n^2 - 2n + 2$. ■

8. HOW MANY? (NON-NEGATIVE INTEGERS) (PRACTICE). Let's look at some numerical examples. The third proof of Theorem 8, in the case $n = 3$, produces the semi-magic square

$$\begin{pmatrix} 34 & 6 & 15 \\ 7 & 47 & 1 \\ 14 & 2 & 39 \end{pmatrix}$$

with magic constant 55, so this matrix cannot be written as a positive integer linear combination of fewer than 5 permutation matrices. But the same is true of the semi-magic square

$$\begin{pmatrix} 1 & 3 & 3 \\ 3 & 2 & 2 \\ 3 & 2 & 2 \end{pmatrix}$$

with magic constant 7. For to account for the entry in the upper left corner, either the identity or (23) must be involved. By symmetry, it doesn't matter which, so let's assume I is a summand. Subtracting I leaves a one in the $(2,2)$ position, which forces involvement of (13), and a one in the $(3,3)$ position, which forces involvement of (12). Subtracting these leaves a matrix with two non-zero entries in each line, so two more matrices are needed; in total, 5.

By brute force, one can show that this result is sharp, that is, that every 3×3 semi-magic square with magic constant less than 7 can be written as a positive integer linear combination of 4 or fewer permutation matrices.

Theorems 7 and 8 imply that every 4×4 semi-magic square can be written as a non-negative integer linear combination of 10 permutation matrices, and that there exist 4×4 semi-magic squares that cannot be written as a non-negative integer linear combinations of 9 permutation matrices. A 4×4 semi-magic square that requires 10 permutations is

$$A = \begin{pmatrix} 5 & 5 & 7 & 14 \\ 11 & 18 & 1 & 1 \\ 10 & 3 & 16 & 2 \\ 5 & 5 & 7 & 14 \end{pmatrix}$$

with magic constant 31; we have been unable to find an example with a smaller magic constant. The proof that this semi-magic square requires 10 permutations reveals a method for producing, for any n, an $n \times n$ semi-magic square that cannot be represented by (that is, written as a non-negative linear combination of) fewer than $n^2 - 2n + 2$ permutation matrices.

Let $A = \sum_{j=1}^{r} a_j Q_j$ with positive integers a_j and permutation matrices Q_j. Since $A(2,3) = 1$, there must be some j such that $Q_j(2,3) = 1$. We may assume $Q_1(2,3) = 1$. Then $a_1 = 1$. Let $A_1 = A - a_1 Q_1$.

Now $A(2,4) = 1$ and $Q_1(2,4) = 0$ (since $Q_1(2,3) = 1$—this is a refinement in the reasoning of the third proof of Theorem 7). So $A_1(2,4) = 1$, and we may assume $Q_2(2,4) = 1$ and $a_2 = 1$. Let $A_2 = A_1 - a_2 Q_2$.

Since $1 \le A_2(3,2) \le 3$, we may assume $Q_3(3,2) = 1$ and $1 \le a_3 \le 3$.

By similar reasoning we find $Q_4(3,4) = Q_5(4,2) = Q_6(4,3) = 1$, $1 \le a_4 \le 2$, $1 \le a_5 \le 5$, and $1 \le a_6 \le 7$. Let $A_6 = A - \sum_{j=1}^{6} a_j Q_j$.

Now comes the tricky part; showing that $A_6(1,1) \ge 1$ (whence $Q_j(1,1) = 1$ for some $j \ge 7$). If $Q_3(1,1) = 1$ then, since Q_3 is a permutation matrix and $Q_3(3,2) = 1$ we must have $Q_3(2,3) = 1$ or $Q_3(2,4) = 1$. Thus, $Q_3(1,1) \le Q_3(2,3) + Q_3(2,4)$. Similarly, $Q_5(1,1) \le Q_5(2,3) + Q_5(2,4)$ and $Q_6(1,1) \le Q_6(2,4) + Q_6(3,4)$. It follows that

$$\begin{aligned}\sum_{j=1}^{6} a_j Q_j(1,1) \le{}& (a_3 Q_3(2,3) + a_5 Q_5(2,3) + a_1)\\ &+(a_3 Q_3(2,4) + a_5 Q_5(2,4) + a_6 Q_6(2,4) + a_2)\\ &+(a_6 Q_6(3,4) + a_4)\\ \le{}& A(2,3) + A(2,4) + A(3,4) = 4.\end{aligned}$$

Since $A(1,1) = 5$, we have established $A_6(1,1) \ge 1$. With the obvious definitions, the same sort of reasoning shows that $A_7(1,2)$, $A_8(1,3)$, and $A_9(1,4)$ are all positive, so $r \ge 10$.

We can prove that any 4×4 square with magic constant 14 or less can be written with fewer than 10 permutation matrices, but we have been unable to close the gap between 14 and 31, or the much larger gaps in our knowledge for $n > 4$.

9. HOW MANY? (SMALL MODULI). Let q be a positive integer, and let A be an $n \times n$ constant line-sum matrix over $\mathbf{Z}/q\mathbf{Z}$. We showed in Section 6 that A can be expressed as a $\mathbf{Z}/q\mathbf{Z}$-linear combination of no more than $n^2 - 2n + 2$ permutation matrices. If q is not too big (relative to n), we can do better.

Theorem 9. *Any $n \times n$ constant line-sum matrix over $\mathbf{Z}/q\mathbf{Z}$ can be written as a $\mathbf{Z}/q\mathbf{Z}$-linear combination of no more than $(q-1)n$ permutation matrices.*

We note that $(q-1)n$ is less than $n^2 - 2n + 2$, provided $q \le n - 1$. We illustrate Theorem 9 with an example before embarking on the proof. Working over $\mathbf{Z}/3\mathbf{Z}$, consider

$$A = \begin{pmatrix} 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 1 & 2 & 2 & 0 \end{pmatrix}. \tag{2}$$

We can construct a semi-magic square A' that is congruent to A (modulo 3), with magic constant $8 = (q-1)n$:

$$A' = \begin{pmatrix} 2 & 2 & 2 & 2 \\ 4 & 1 & 1 & 2 \\ 1 & 3 & 3 & 1 \\ 1 & 2 & 2 & 3 \end{pmatrix}.$$

Trivially, A' can be written as a sum of 8 permutation matrices; this serves to express A as a $\mathbf{Z}/3\mathbf{Z}$-linear combination of 8 permutation matrices.

Proof of Theorem 9: Let A be an $n \times n$ constant line-sum matrix over $\mathbf{Z}/q\mathbf{Z}$. We may view the entries of A as integers a_{ij} satisfying $0 \le a_{ij} \le q-1$. Working now in $\mathbf{Z}$, let the maximal line sum in A be m; note that $m \le (q-1)n$. We now construct a semi-magic square A', congruent, entrywise, to A (modulo q), with magic constant m. Choose any row of A whose entries do not add up to m (if there is no such row, A is already semi-magic), and any column of A whose entries do not add up to m. Where the chosen row and column intersect, add to the entry a large enough multiple of q to bring the larger of the row and column sums up to m. This does not change the congruence class of the entry (modulo q), and it decreases by at least one the number of lines with line-sum not equal to m. After at most $2n-2$ applications of this procedure we arrive at a semi-magic square, A'.

Now A' is a semi-magic square with magic constant m, so it can certainly be written as a sum of m permutation matrices. As corresponding entries in A' and A are congruent (modulo q), the same m permutation matrices sum to A when viewed over $\mathbf{Z}/q\mathbf{Z}$. Since $m \le (q-1)n$, we are done. ■

In the case $q = 2$, Theorem 9 is best possible, since it is clear that the $n \times n$ all-ones matrix requires n permutation matrices. In other cases, we can often do better; if q is not a prime, we can always do better. It helps to introduce some notation here. Let $\beta(A,q)$ denote the least r such that A can be written as a $\mathbf{Z}/q\mathbf{Z}$-linear combination of r permutation matrices, and let $\beta(n,q)$ denote the maximum value of $\beta(A,q)$ over all $n \times n$ constant line-sum matrices A. In this notation, Theorem 9 says $\beta(n,q) \le (q-1)n$.

Theorem 10. *Let s and t be integers, and let A be an $n \times n$ constant line-sum matrix over $\mathbf{Z}/st\mathbf{Z}$. Then $\beta(A,st) \le \beta(A,s) + \beta(n,t)$.*

Proof: Let $\beta(A,s) = k$, so $A = \Sigma_1^k c_j P_j + sA_1$ for some integers $c_1, \ldots, c_k$, some permutation matrices $P_1, \ldots, P_k$, and some constant line-sum matrix A_1. Then we see that $A_1 = \Sigma_1^l d_j Q_j + tA_2$ for some integers $d_1, \ldots, d_l$, some permutation matrices $Q_1, \ldots, Q_l$, and some constant line-sum matrix A_2, with $l \le \beta(n,t)$. Then

$$A \equiv \sum^{k} c_j P_j + \sum^{l} s d_j Q_j \pmod{st},$$

and $k + l \le \beta(A,s) + \beta(n,t)$. ■

Corollary 11. *Let the factorization of q into powers of distinct primes be $q = p_1^{a_1} \cdots p_r^{a_r}$. Then*

$$\beta(n,q) \le \sum_1^r a_j \beta(n,p_j) \le \sum_1^r a_j (p_j - 1) n. \tag{3}$$

If q is not a prime then (3) is always an improvement over the bound in Theorem 9. We can often make a small improvement on the bound (3), even for prime q. Rather than state the result in its full (and somewhat tedious) generality, we illustrate its application to 4×4 matrices over $\mathbf{Z}/3\mathbf{Z}$ by establishing that $\beta(4,3) \le 7$; Theorem 9 allows us to conclude only that $\beta(4,3) \le 8$. Let A be any 4×4 constant line-sum matrix over $\mathbf{Z}/3\mathbf{Z}$ that, when viewed as an integer matrix, has maximal line-sum 8; for example, the matrix (2). Then $2A$ has all line-sums

congruent to 1 (mod 3), thus, maximal line-sum at most 7 (when viewed as an integer matrix). In our example,

$$2A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 \\ 2 & 0 & 0 & 2 \\ 2 & 1 & 1 & 0 \end{pmatrix}.$$

By the procedure of the proof of Theorem 9, $2A$ can be expressed, over $\mathbf{Z}/3\mathbf{Z}$, as a sum of 7 permutation matrices. Multiplication by 2 yields an expression for A as a $\mathbf{Z}/3\mathbf{Z}$-linear combination of 7 permutation matrices, whence $\beta(4,3) \leq 7$.

With a bit more work, we can actually prove $\beta(4,3) = 6$. For it follows from the work of Marcus and Minc [4] that if B is a 4×4 semi-magic square with magic constant 7, then there is a permutation matrix P such that $B - 2P$ has non-negative entries. Since $B - 2P$ is a semi-magic square with magic constant 5, B is a positive integer linear combination of 6 or fewer permutation matrices. Thus, the number of permutation matrices necessary to represent a 4×4 constant line-sum matrix over $\mathbf{Z}/3\mathbf{Z}$ is at most 6, which is best possible:

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \end{pmatrix}$$

cannot be written as a $\mathbf{Z}/3\mathbf{Z}$-linear combination of fewer than 6 permutation matrices (exercise for the reader). The general question of evaluating $\beta(n,q)$ appears to be very intricate.

10. APPENDIX. We present a result about vector spaces that is somewhat technical, together with two useful corollaries. We would like to thank Bruce Reznick for suggestions that improved the exposition in the proof of this lemma.

Lemma 12. *Let V be a vector space over a field F. Let $v_1, \dots, v_d$ and z be in V, and let $W_1, \dots, W_m$ be subspaces of V. Assume that no W_i contains the subspace V_0 generated by $\{v_1, \dots, v_d\}$. Let $S \subseteq F$ be any set with $m + 1$ or more elements. Then there is a vector v in V that can be written as $v = a_1 v_1 + \cdots + a_d v_d + z$ with each a_i in S, but v is not in $W_1 \cup \cdots \cup W_m$.*

Corollary 13. *No vector space V over an infinite field F is a finite union of proper subspaces.*

Proof: Let $W_1, \dots, W_m$ be proper subspaces of V. Choose v_i in V such that v_i is not in W_i for $1 \leq i \leq m$. Now apply Lemma 12, with $S = F$ and $z = 0$. ■

Corollary 14. *For every n there is an $n \times n$ constant line-sum matrix with non-negative rational entries that is not a rational linear combination of $n^2 - 2n + 1$ permutation matrices.*

Proof: In Lemma 12, let F be the rationals, and let S be the non-negative rationals. Take $d = n^2 - 2n + 2$, and let $v_1, \dots, v_d$ be a linearly independent set of permutation matrices. Let V be the span of $\{v_1, \dots, v_d\}$, which is the space of all $n \times n$ constant line-sum matrices with rational entries. Let $W_1, \dots, W_m$ be the subspaces generated by sets of $n^2 - 2n + 1$ permutation matrices—one subspace

for each set of permutation matrices. Lemma 12 ensures that there exist $a_1, \ldots, a_d$, all non-negative rationals, such that $v = a_1 v_1 + \cdots + a_d v_d$ is not in any W_i. This v is a constant line-sum matrix with non-negative rational entries, and is not a rational linear combination of $n^2 - 2n + 1$ permutation matrices. ■

Proof of Lemma 12. We may assume that $v_1, \ldots, v_d$ are linearly independent, for, if $v_1, \ldots, v_r$ are linearly independent, and $v_{r+1}, \ldots, v_d$ are dependent on $v_1, \ldots, v_r$, we may choose $a_{r+1}, \ldots, a_d$ arbitrarily from S, let $z' = a_{r+1} v_{r+1} + \cdots + a_d v_d + z$, and find a vector v that can be written as $v = a_1 v_1 + \cdots + a_r v_r + z'$.

For each j, let $X_j = W_j \cap V_0$. Then $X_1, \ldots, X_m$ are proper subspaces of V_0. We may assume that S has exactly $m + 1$ elements, and let $T = \{\sum_{i=1}^{d} a_i v_i + z : a_i \in S\}$, so T has cardinality $(m+1)^d$. We wish to conclude that T is not contained in $X_1 \cup \cdots \cup X_m$.

In fact, we prove that $\#(X_j \cap T) \le (m+1)^{d-1}$, from which it follows that

$$\begin{aligned}\#((X_1 \cup \ldots \cup X_m) \cap T) &\le \sum_j \#(X_j \cap T) \le m(m+1)^{d-1} < (m+1)^d \\ &= \#(T).\end{aligned}$$

For, suppose $\#(X_1 \cap T) > (m+1)^{d-1}$. Then for each k, $1 \le k \le d$, the pigeonhole principle implies that there exist $c_1, \ldots, c_{k-1}, c_{k+1}, \ldots, c_d$ in S such that $c_1 v_1 + \cdots + b v_k + \cdots + c_d v_d + z$ is in X_1 for two distinct elements b of S, say, $b = b_1$ and $b = b_2$. Then $(b_2 - b_1) v_k$ is in X_1, hence v_k is in X_1. But this is true for each k, contradicting the hypothesis that X_1 is a proper subspace of V_0. The same argument applies to each X_j. ■

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The Isoperimetric Problem on Surfaces

Hugh Howards, Michael Hutchings, and Frank Morgan

1. INTRODUCTION. The isoperimetric problem on a surface is to enclose a given area with the shortest possible curve. The classical isoperimetric theorem asserts that in the plane the unique solution is a circle. On curved surfaces the isoperimetric problem is harder and much remains open. Even on the simplest paraboloid the "obvious" solution was proved only in 1996 by Benjamini and Cao ([**2**, Thms. 5, 8]; see also [**24**, Prop. 7], [**22**, Thm. 3.1], [**30**, Thm. 1], [**29**], [**26**]):

Theorem 1.1 (Benjamini and Cao). *The unique least-perimeter way to enclose given area in the paraboloid of revolution*

$$P = \{z = x^2 + y^2\} \subset \mathbf{R}^3 \tag{1.1}$$

is a horizontal circle $\{z = c\}$.

This article gives our three favorite proofs of the classical isoperimetric theorem in the plane and then presents some recent results on other surfaces, including a new proof for the paraboloid. Section 2 uses an amazingly simple symmetry argument to show that a nice minimizer must be a circle. Unfortunately this approach needs to assume that a nice minimizer exists. Section 3 gives a very simple, complete proof without assuming a nice minimizer exists, following the undergraduate thesis of Howards [**15**]. Section 4 provides another complete proof, a slight twist on a magical proof of Gromov [**10**].

In general surfaces the existence of a nice, one-component perimeter-minimizing curve has been astonishingly problematic. Fortunately a relatively easy approach is now available from [**12**], as explained in Section 5. One has to allow the curve to bump up against itself.

Sections 6–8 solve the isoperimetric problem for cylinders, cones, flat tori, and Klein bottles. Section 9 treats the paraboloid and certain other surfaces of revolution. Section 10 discusses hyperbolic surfaces.

This work was partly inspired by a more difficult question we heard from J. C. C. Nitsche about the soap film between a large wire boundary and a small, moveable loop of thread. The thread wants to position itself to minimize the area of the soap film outside it. If the thread were constrained to lie in a fixed surface bounded by the wire (which unfortunately is not the case), then the thread would want to be an isoperimetric curve in that surface.

Osserman [**23**] provides a marvelous survey on the isoperimetric inequality.

2. THE CIRCLE IN THE PLANE, ASSUMING SMOOTH EXISTENCE. We assume that there is a compact minimizer C among smooth curves of finitely many components and enclosed area π, and use symmetry to prove it must be a single round unit circle; existence is a nontrivial assumption, a fact overlooked by some early workers. The proof uses a symmetry argument we heard from Brian White and Luen-fai Tam, who thought it originated with Blaschke (see [**9**, Thm. 3.4], [**17**, Thm. 5.3], and [**16**, §2]); we have been unable to trace its origin and would be grateful to anyone who could help.

Suppose C is not a round circle. Take a horizontal line splitting the enclosed area in half. Each half must have the same length, or the shorter half, together with its reflection, would be shorter than C. Replacing C by half plus its reflection if necessary, we may assume that C is symmetric across the horizontal line. Similarly we may assume that C is symmetric across a vertical line. We may assume the lines meet at the origin. Now C is symmetric under the composition of the two reflections, i.e., under 180-degree rotation around the origin. Hence every line through the origin splits the area in half. C must meet every line through the origin orthogonally; otherwise, one half of C, together with its reflection, would not be convex, and its convex hull would have less perimeter and more area. It follows that C consists of circles about the origin. A single circle is best. We conclude that the original C is a round circle.

This argument can be generalized to prove that a round hypersphere is perimeter-minimizing for given volume in $\mathbf{R}^n$, in the round sphere $\mathbf{S}^n$, and in hyperbolic space $\mathbf{H}^n$. More generally, it shows that a minimizing cluster of k bubbles enclosing $k < n$ prescribed volumes in $\mathbf{R}^n$ has $\mathbf{O}(n - k + 1)$ symmetry, assuming known but difficult existence and regularity [**16**, Thm. 2.6]. It played an essential role in the recent proof by Hass, Hutchings, and Schlafly of the equal volumes case of the still open Double Bubble Conjecture, which says that the familiar standard double soap bubble is the least-area way to enclose and separate two given volumes of air ([**11**], [**16**], [**14**], [**18**], [**13**]).

3. THE CIRCLE IN THE PLANE, WITHOUT ASSUMING EXISTENCE. To prove that the circle is perimeter-minimizing (but not necessarily uniqueness), by approximation it suffices to show that the shortest n-gon enclosing given area is the regular n-gon. In his undergraduate thesis, Howards [**15**] gave the following geometric proof free of variational calculus, including ideas that we have since traced back to Zenodorus about 200 BC, Steiner in 1838 ([**27**], [**28**, p. 105 and Fig. 6]), and Courant and Hilbert [**6**, p. 166]; see the interesting "A history of the classical isoperimetric problem" by Porter [**25**] and Bonnesen and Fenchel [**5**, §57].

By compactness, there is a shortest n-gon in the $2n$-dimensional space of vertices. It is convex. Consider two adjacent sides, which determine a triangle, and the line L through the common vertex and parallel to the third side of the triangle. These two sides must constitute the shortest path to L and back, since all such constructions yield triangles of the same area. The first side, together with the reflection of the second across L, must form a straight line. Hence the two sides have the same length. Therefore the n-gon is equilateral.

To prove that the equilateral n-gon is regular, we begin with n even. For opposite vertices P, Q, the line PQ must have the same area above as below, or a reflection of the larger half would enclose more area (or, scaled down, the same area with less length). For an intermediate vertex M, the angle PMQ must be 90°, or replacing it with a 90° angle and reflecting as in Figure 3.1 would increase the area enclosed. Therefore the n-gon is inscribed in a circle and must be regular.

Finally suppose n is odd. A regular $2n$-gon comes from putting little triangles on the sides of the regular n-gon. If a perimeter-minimizing n-gon, known to be equilateral, had more area than a regular n-gon with the same sides, putting those little triangles on its sides would yield a non-regular $2n$-gon with more area than the regular $2n$-gon, the final contradiction.

This completes the proof that the circle is perimeter minimizing. In fact, now that we know that a minimizer exists, we can use the above arguments to prove

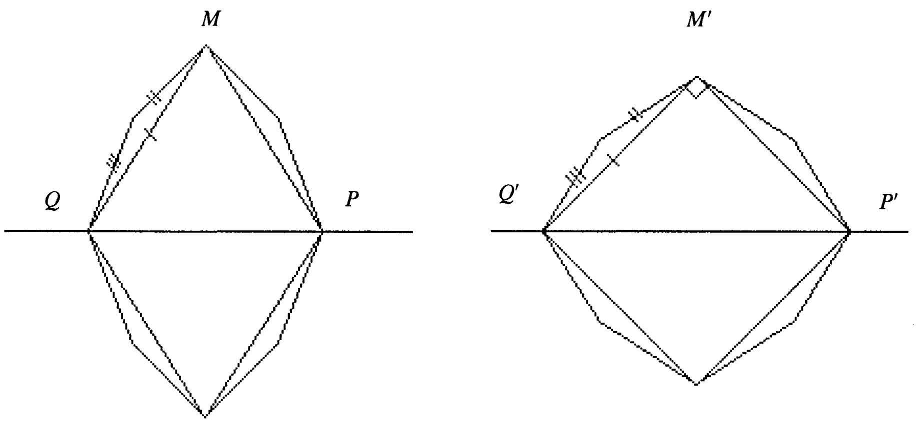

Figure 3.1. The angle PMQ must be 90°, or replacing it with a 90° angle and reflecting would increase the area enclosed.

uniqueness. Consider any minimizer. It must be convex. As above, a line bisecting its perimeter must bisect the area, and any inscribed angle must be 90°. Therefore, the minimizer must be a circle.

4. THE CIRCLE IN THE PLANE, ANOTHER PROOF WITHOUT ASSUMING EXISTENCE. Gromov ([**10**]; see [**3**, 12.11.4] or [**20**, 10.5]) gave a proof of the isoperimetric theorem in $\mathbf{R}^n$ by direct comparison. The strategy in $\mathbf{R}^2$, for example, is to find a vectorfield v on any competing region R of area π with smooth boundary C and outward unit normal $\mathbf{n}$ such that

$$\operatorname{div}(v) \geq 2, \tag{4.1}$$

$$v \cdot \mathbf{n} \leq 1, \tag{4.2}$$

with equality everywhere only if R is a disc. If such a v can be found, then the isoperimetric inequality follows immediately from Stokes' theorem:

$$\operatorname{length}(C) \geq \int_C v \cdot \mathbf{n} = \int_R \operatorname{div}(v) \geq 2\operatorname{area}(R) = 2\pi,$$

with equality only if R is a disc.

The Gromov proof finds such a v by a very clever construction, but the resulting v is not canonical. We now show that there is a canonical such v when $n = 2$.

The canonical v is the negative of the *gravitational field* induced by a substance of constant density filling the region R. More precisely,

$$v(x) = \frac{1}{\pi}\int_{y \in R} \frac{x - y}{|x - y|^2}\, dy.$$

By the two-dimensional analog of Gauss's law, $\operatorname{div}(v) = 2$ in R, so it now suffices to prove (4.2).

Fix a point x in the boundary, and choose polar coordinates (r, θ) around x so that $\mathbf{n}$ points in the direction $\theta = \pi$. Then

$$v(x) \cdot \mathbf{n} = \frac{1}{\pi}\int_{y \in R} \frac{\cos\theta}{r}\, dy.$$

Since the area of R is fixed, this integral is maximized if we put the points of R where $(\cos\theta)/r$ is largest. The level sets of $(\cos\theta)/r$ are circles tangent to C at x, with smaller circles giving larger values of $(\cos\theta)/r$. So clearly a disc of the given area uniquely maximizes the integral, completing the proof.

5. PROOF OF EXISTENCE OF NICE LEAST-PERIMETER ENCLOSURES. In the Euclidean plane and in other special cases where all candidates can be convexified, the existence of a (convex) region of least perimeter and prescribed area follows from Blaschke's selection theorem ([**4**, p. 38], [**8**, Chapt. 4]). A general smooth Riemannian surface S requires a more general argument. There must be some hypothesis to prevent the solution from disappearing to infinity as in Figure 5.1. Suppose for now that S is a compact surface, perhaps with convex boundary.

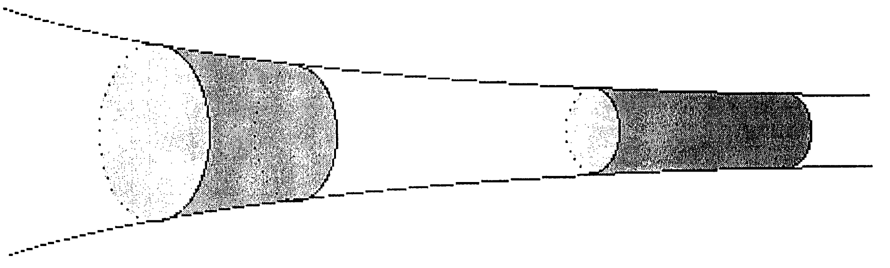

Figure 5.1. In the surface of revolution generated by $y = 1/x$, for any given area, there is a sequence of annuli disappearing to infinity with perimeters going to 0.

For the moment we restrict to images of the unit circle parametrized by arclength. Later we consider curves of several components. Then compactness properties of Lipschitz functions (Ascoli-Arzela Theorem) immediately produce a minimizer. The only problem is that in theory the limit might bump up against itself too wildly to permit the standard variational argument that it has constant geodesic curvature. The solution may actually bump up against itself, as in the cylinder of Figure 5.2. This technical difficulty delayed for 75 years the completion of Poincare's proof that every smooth convex sphere contains a simple closed geodesic. In 1982 C. Croke [**7**] gave a complete proof by minimizing a combination of length and energy in a class of piecewise geodesic curves.

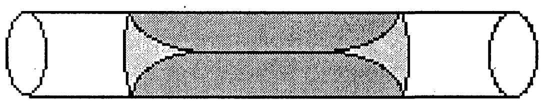

Figure 5.2. Some least perimeter enclosures on the cylinder bump up against themselves.

More recently, Hass and Morgan ([**12**]; see also [**22**, Lemma 2.2]) have provided a very simple approach to more general existence and regularity using local convexification. Away from the boundary of S, a minimizing enclosure is an embedded curve of constant geodesic curvature κ_0, except possibly for finitely many geodesic arcs or isolated points where it bumps up against itself but remains C^1. Even at the boundary of S the curve remains C^1 and the geodesic curvature

satisfies $\kappa \leq \kappa_0$ (weakly). If for bounded area the curve is allowed a large number of components enclosing disjoint regions, no curve bumps itself on the inside and $\kappa \leq \kappa_0$ everywhere. If the curve is allowed several nested components enclosing multiply connected regions, it never bumps itself. A word on the proof: if local convexification causes two pieces of curve to cross, the longer one is rerouted along the shorter. This process reduces length unless the curves were convex to begin with. Given convexity, standard variational arguments prove the rest.

In many noncompact manifolds, such as the Euclidean or hyperbolic plane, one can work inside a large convex set. Hyperbolic surfaces and other surfaces can have thin cusps to infinity with nonconvex truncations, but as long as the area of the cusp is finite, any sequence of curves going off to infinity has area going to 0 and may be discarded.

Existence and some regularity hold as well for clusters in $\mathbf{R}^2$ (enclosing and separating several regions of prescribed areas [**21**]) and in general dimensions by the techniques of geometric measure theory [**19**, Chapt. 13]. In higher dimensions you cannot hope to prescribe the topological type; for example, regions connected by thin tubes can disconnect in the limit. Even for curves in the plane, such general techniques do not have the topological control we need.

6. CIRCULAR CYLINDERS. *On the cylinder* $\{x^2 + y^2 = a^2\} \subset \mathbf{R}^3$, *the least-perimeter enclosure of area* A *is a small (round) circle for* $A \leq 4\pi a^2$ *and two horizontal circles for* $A \geq 4\pi a^2$.

Proof: We know that any solution consists of closed curves of constant curvature. If one curve is homotopically trivial and hence is a small round circle, it is the only one, or it could be translated to touch another and contradict regularity. If all the curves are homotopically nontrivial, there must be at least two of them to enclose area, and two horizontal circles are best. The transition occurs when the circumference of the small circle $2\sqrt{\pi A}$ equals the length of two horizontal circles $4\pi a$, i.e., $A = 4\pi a^2$.

7. FLAT TORI AND KLEIN BOTTLES (Howards [**15**, Thm. 3.1]). *Let* S *be a flat torus or Klein bottle with shortest closed geodesic of length* a. *Given* $0 < A <$ *area* S, *the least-perimeter region of area* A *is*

(1) *a circular disc if* $0 < A \leq a^2/\pi$;
(2) *a band (possibly Möbius) with geodesic boundary if* $a^2/\pi \leq A \leq$ *area* $S - a^2/\pi$;
(3) *the complement of a circular disc if area* $S - a^2/\pi \leq A \leq$ *area* S.

Proof: Any solution consists of closed curves of constant curvature. As in the argument in Section 6, if one is homotopically trivial and therefore a small round circle, it is the only one or it could be translated to touch another and contradict regularity. If all the components are nontrivial, for any given area the perimeter is uniquely minimized by a single geodesic band with perimeter $2a$. The transitions between types occur when the circle has circumference $2a$.

Remark. The round sphere and round projective plane may be treated by similar arguments [**15**, Thms. 4.1, 5.1] or by the methods of Section 9 [**22**, Thms. 3.1, 3.3].

8. CIRCULAR CONES. *On the circular cone* $\{z = a\sqrt{x^2 + y^2}\} \subset \mathbf{R}^3$, *the least-perimeter enclosure of area* A *is a horizontal circle.*

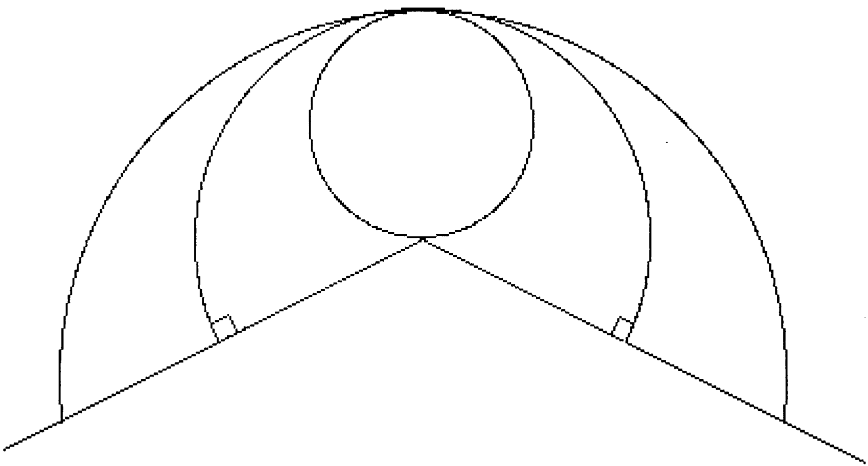

Figure 8.1. A constant-curvature curve about the vertex of the cone must be a circle of smaller circumference than a planar circle of the same area.

Proof: If any component does not encircle the vertex, it must be the only component (or it could be translated to touch another component, contradicting regularity), and hence it must be a circle of length $2\sqrt{\pi A}$. Consider a constant-curvature curve that encircles the vertex. It must be symmetric about the line through the vertex and a point most distant from the vertex, as in Figure 8.1, so it must be a horizontal circle. Clearly a single horizontal circle would have less perimeter than several. Since one horizontal circle has length less than $2\sqrt{\pi A}$, it must be the minimizer.

Remark. Actually for a simply-connected domain D on a surface with Gauss curvature K, the perimeter L satisfies

$$L^2 \geq 4\pi A - 2A\int_D \max\{K, 0\},$$

with equality for the singular limit case of the cone [**23**, Thm. 4.3].

9. THE PARABOLOID AND OTHER SURFACES OF REVOLUTION. Sections 9 and 10 provide some new examples. The following theorem and corollary include the paraboloid. The proof integrates the Gauss-Bonnet theorem.

Theorem 9.1 ([**24**, Prop. 7], [**22**, Thm. 2.1], [**29**]). *Consider the plane with smooth, rotationally symmetric, complete metric such that the Gauss curvature is a strictly decreasing function of the distance from the origin. Then the unique length-minimizing simple closed curve enclosing a given area is a circle centered at the origin.*

By Section 5, inside a surface of finite area or inside a large convex ball B, for bounded area, there is a minimizer among C^1 curves of $m \leq m_0$ components, enclosing m disjoint discs. Away from ∂B, it is an embedded curve of constant geodesic curvature κ_0, except possibly for finitely many geodesic arcs or isolated points where it bumps up against itself. If m_0 is large, the curvature $\kappa \leq \kappa_0$ everywhere.

A short proof of the following standard technical lemma is in [**20**, §9.7, p. 112] (cf. [**22**, Lemma 2.3]). The idea is that the rate of change of the perimeter is essentially the geodesic curvature, which is controlled by the Gauss-Bonnet theorem.

Lemma 9.2. *Let $L(A)$ denote the least perimeter of $m \leq m_0$ discs of total area A. Then $L(A)$ is differentiable almost everywhere and*

$$L(A)^2 \geq 2\int_0^A LL'.$$

Proof of Theorem 9.1 [**22**]: In the surface or inside a large convex ball, for m_0 large, let $L(A)$ denote the length of a shortest curve of $m \leq m_0$ components enclosing m disjoint discs of total area A. First we claim that if L is differentiable at A, then $L'(A)$ is the geodesic curvature κ_0. One geometric interpretation of geodesic curvature is the rate of change of length with respect to area under perturbations of the given minimizer [**20**, Chapt. 2]. Hence for $\Delta A > 0$, the new minimizers must do at least as well as perturbations of the old one, and $L'(A) \leq \kappa_0$. Similarly for $\Delta A < 0$, $-L'(A) \leq -\kappa_0$. Therefore $L'(A) = \kappa_0$.

Now Gauss-Bonnet tells us that the total Gauss curvature of the enclosed region equals

$$2\pi m - \int \kappa \geq 2\pi - L(A)\kappa_0 = 2\pi - L(A)L'(A).$$

Let $G(A)$ denote the total Gauss curvature of a disc of area A centered at the origin. Since the Gauss curvature is a decreasing function of radius, any other region with the same area must have less total Gauss curvature. So we have

$$2\pi - L(A)L'(A) \leq G(A), \tag{9.1}$$

and

$$L(A)L'(A) \geq 2\pi - G(A).$$

By Lemma 9.2, integration from $A = 0$ to A_1 yields

$$L(A)^2 \geq 2\int L(A)L'(A) \geq 4\pi A_1 - 2\int G(A). \tag{9.2}$$

This inequality is sharp for a circle centered at the origin, as we can see by integrating the Gauss-Bonnet formula for circles centered at the origin with area A from 0 to A_1. Hence equality holds in (9.2), L is differentiable everywhere, and equality holds in (9.1). Therefore a minimizer encloses Gauss curvature $G(A)$ and must be a circle about the origin.

The following general extension to several, perhaps multiply connected regions is deduced in [**22**, Thm. 3.1]. Here we give a proof for the easy case of positive Gauss curvature, which includes the paraboloid.

Corollary 9.3. *Among unions of disjoint, perhaps multiply connected regions, a perimeter minimizer exists and is a (round) disc, disc complement, or annulus about the origin. If the Gauss curvature is positive or the total Gauss curvature of every compact region is less than 2π, then the minimizer is a disc.*

Proof for the case of positive Gauss curvature: By the Gauss-Bonnet theorem, the perimeter $P(r)$ and geodesic curvature $\kappa(r)$ of a circle about the origin of radius r

bounding a disc D satisfy

$$P' = \kappa P = 2\pi - \iint_D K. \tag{9.3}$$

The total Gauss curvature is at most 2π, since otherwise eventually $P' \leq -a < 0$ and P hits 0. By (9.3), κ is positive and decreasing. Consider any collection of simple closed curves enclosing area A_1. By discarding any curves inside others, enclose area $A_2 \geq A_1$. By Theorem 9.1, each curve alone would be shortest if a circle about the origin. Since $\kappa = dP/dA$ is decreasing, one single circle about the origin is best. Since $A_1 \leq A_2$, the circle of area A_1 is best of all.

10. HYPERBOLIC MANIFOLDS. We consider geometrically finite complete hyperbolic surfaces (curvature -1). Such surfaces may be compact or have finitely many ends: cusps (with exponentially shrinking thickness and finite area) or flared ends (asymptotic to the hyperbolic plane).

Theorem 10.1 [**1**, Thm. 2.2]. *Let S be a hyperbolic surface. For given area $0 < A <$ area(S), a perimeter-minimizing system of embedded rectifiable curves bounding a region of area A consists of curves of the same constant curvature of one of four types:*

(I) *a circle,*
(II) *horocycles around cusps,*
(III) *two "neighboring curves" at constant distance from a geodesic, bounding an annulus or complement,*
(IV) *geodesics or single neighboring curves.*

The total perimeter L satisfies

$$L \leq \sqrt{A^2 + 4\pi A}, \tag{10.1}$$

with equality for a circle of area A. If S has at least one cusp, then cases (I) *and* (III) *do not occur and $L \leq A$; if moreover $A < \pi$, then a minimizer consists of horocycles bounding neighborhoods of an arbitrary collection of cusps and has perimeter $L = A$.*

Proof sketch: The constant-curvature curves on a hyperbolic surface are circles bounding discs ($\kappa > 1$) or the complement ($\kappa < -1$), horocycles around cusps ($\kappa = 1$) or the complement ($\kappa = -1$), and constant-curvature curves around necks ($|\kappa| < 1$, including the geodesics around the middle of necks with $\kappa = 0$).

A minimizer cannot have more than one circle, since sliding one until it hits another (or itself) would contradict regularity. Since for other types, $dL/dA = \kappa$ is less than it is for a circle, (10.1) always holds, and there is an $A_0 \geq 0$ such that if $A < A_0$ the minimizer is a circle, while if $A > A_0$ it is not a circle and (for $\Delta A > 0$)

$$\Delta L/\Delta A < 1. \tag{10.2}$$

Now a computation shows that an annulus (or complement) as in Case (III) must occur alone, or an operation such as discarding it would contradict (10.2). Therefore the minimizer falls into one of the four asserted cases.

Henceforth assume S has a cusp. Case (I) cannot occur, because sliding the circle out the cusp until it hits itself would contradict regularity. Hence the minimizer always has $|\kappa| \leq 1$, and always $L(A) \leq A$. A computation shows that Case (III) cannot occur.

Finally assume $A < \pi$. We claim *there is no minimizer with* $-1 \leq \kappa < 1$ *and length* $L \leq A$, so $-A + \kappa L < 0$. Otherwise, applying Gauss-Bonnet to the enclosed region yields

$$2\pi\chi = -A + \kappa L < 0,$$

$$\chi \leq -1,\ -A + \kappa L \leq -2\pi,\ \kappa L \leq -\pi,\ \kappa < 0, L \geq \pi > A, \text{ a contradiction.}$$

The remaining possibilities, systems of curves with $\kappa = 1$, consist of horocycles bounding cusp neighborhoods. Since $\kappa = 1$, as you slide a horocycle out a cusp $dL/dA = 1$, and therefore its length equals the area of the cusp neighborhood. By the claim, such systems remain minimizing as long as they exist, either for all $A < \pi$ or until they bump up against themselves at some $A_1 < \pi$. If one bumps, by regularity the minimizer has perimeter less than A_1, contradicting the claim and proving the theorem.

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The infinitude of the primes
Is the subject of plenty of rhymes,
But we can't begin
To prove there's a twin
An infinite number of times.

Contributed by Peter Rosenthal, University of Toronto

What Is a Closed-Form Number?

Timothy Y. Chow

1. INTRODUCTION. When I was a high-school student, I liked giving *exact* answers to numerical problems whenever possible. If the answer to a problem were $2/7$ or $\pi\sqrt{5}$ or arctan 3 or $e^{1/e}$, I would always leave it in that form instead of giving a decimal approximation.

Certain problems frustrated me because there did not seem to be any way to express their solutions exactly. For example, consider the following problems.

Question 1. The equation

$$x + e^x = 0 \tag{1.1}$$

has exactly one real root; call it R. Is there a closed-form expression for R?

Question 2. The equation

$$2x^5 - 10x + 5 = 0 \tag{1.2}$$

has five distinct roots r_1, r_2, r_3, r_4, and r_5. Are there closed-form expressions for them?

Questions like this seemed to have a negative answer, but I continued hoping that the answer was yes, and that I just did not know enough mathematics yet.

In college I learned about Galois theory, and that the Galois group of (1.2) is S_5 [7, Section 5.8]. So the r_i are provably not expressible in terms of radicals. But although this probably should have satisfied me, it did not. Consider the equation

$$x^4 - (6\sqrt{3})x^3 + 8x^2 + (2\sqrt{3})x - 1 = 0.$$

Its roots are $\tan(\pi/15)$, $\tan(4\pi/15)$, $\tan(7\pi/15)$, and $\tan(13\pi/15)$. These seemed to me to be perfectly good closed-form expressions. Although in this particular case the roots could also be expressed in terms of radicals, it seemed to me that there might exist algebraic numbers that were not expressible using radicals but that could still be expressed in closed form—say, using trigonometric or exponential or logarithmic functions. So as far as I was concerned, Galois theory was not the end of the story.

When students ask for a closed-form expression for $\int \exp(x^2)\,dx$, we all know the standard answer: the given function is not an elementary function. Curiously, though, Question 1 (as well as Question 2, if you accept my dissatisfaction with the Galois-theoretic answer) does not seem to have a standard answer that "everybody knows." At most we might mutter vaguely that (1.1) is a "transcendental equation," but this is not very helpful.

This nonexistence of a standard answer to such a simple and common question seems almost scandalous to me. The main purpose of this paper is to eliminate this scandal by suggesting a precise definition of a "closed-form expression for a number." This will enable us to restate Questions 1 and 2 precisely, and will let us

see how they are related to existing work in logic, computer algebra, and transcendental number theory. My hope is that this definition of a closed-form expression for a number will become standard, and that many readers will be lured into working on the many attractive open problems in this area.

2. FROM ELEMENTARY FUNCTIONS TO EL NUMBERS. How can we make Questions 1 and 2 precise? Our first inclination might be to turn to the notion of an *elementary function*. Recall that a function is *elementary* if it can be constructed using only a finite combination of constant functions, field operations, and algebraic, exponential, and logarithmic functions. This class of functions has been studied a great deal in connection with the problem of symbolic integration or "integration in finite terms" **[4]**, and it does a rather good job of capturing "high-school intuitions" about what a closed-form expression should look like. For example, in Question 1 above, it turns out that $R = -W(1)$, where W, the *Lambert W function* **[6]**, is the (multivalued) function defined by the equation

$$W(x)e^{W(x)} = x.$$

But since W is not an elementary function **[5]**, this is not an answer that would satisfy most high-school students. Similarly, if we allow various special functions—e.g., elliptic, hypergeometric, or theta functions—then we can explicitly express the r_i in Question 2, or indeed the roots of any polynomial equation, in terms of the coefficients; see **[3]** and **[10]**. But this again feels unsatisfactory because these special functions are not elementary.

The concept of an elementary function is certainly on the right track, but observe that what we need for Questions 1 and 2 is a notion of a closed-form *number* rather than a closed-form *function*. The distinction is important; we cannot, for example, simply define an "elementary number" to be any number obtainable by evaluating an elementary function at a point, because all constant functions are elementary, and this definition would make all numbers elementary. Furthermore, even if a function (such as W) is not elementary, it is conceivable that each particular value that it takes ($W(1)$, $W(2), \dots$) could have an elementary expression, but with different-looking expressions at different points. These difficulties can probably be circumvented with a little work, but we take a different tack; instead of trying to define closed-form numbers *in terms of* elementary functions, we give an *analogous* definition.

We mention one more technical point. By convention, all algebraic functions (i.e., functions that satisfy a polynomial equation with polynomial coefficients) are considered to be elementary, but this is not suitable for our purposes. Intuitively, "closed-form" implies "explicit," and most algebraic functions have no simple explicit expression. So the set of *purely transcendental elementary functions* is a better prototype for our purposes than the set of elementary functions. "Purely transcendental" simply means that the word "algebraic" is dropped from the definition.

With all these considerations in mind, we propose the following fundamental definition.

Definition. A subfield F of $\mathbb{C}$ is *closed under* exp *and* log if (1) $\exp(x) \in F$ for all $x \in F$ and (2) $\log(x) \in F$ for all nonzero $x \in F$, where log is the branch of the natural logarithm function such that $-\pi < \operatorname{Im}(\log x) \leq \pi$ for all x. The field $\mathbb{E}$ of *EL numbers* is the intersection of all subfields of $\mathbb{C}$ that are closed under exp and log.

Before discussing $\mathbb{E}$, let us make some remarks about terminology. It might seem more natural to call $\mathbb{E}$ the field of *elementary numbers*, but unfortunately this term is already taken. It seems to have been first used by Ritt [**18**, p. 60]. By analogy with elementary functions, Ritt thought of elementary numbers as the smallest *algebraically closed* subfield $\mathbb{L}$ of $\mathbb{C}$ that is closed under exp and log. It so happens that terminology has evolved since Ritt, so that "elementary numbers" are now numbers that can be specified *implicitly* as well as explicitly by exponential, logarithmic, and algebraic operations, and $\mathbb{L}$ is now sometimes called the field of *Liouvillian numbers* [**16**]. But either way, calling $\mathbb{E}$ the field of elementary numbers would conflict with existing usage. The "EL" in the term "EL number" is intended to be an abbreviation for "Exponential-Logarithmic" as well as a diminutive of "Elementary"; it reminds us that $\mathbb{E}$ is a subfield of the elementary numbers.

I should also remark that I am certainly not the first person ever to have considered the field $\mathbb{E}$, but it has received surprisingly little attention in the literature and nobody seems to have lobbied for it as a fundamental object of interest, which in my opinion it is (as illustrated by my temerity in using "blackboard bold" for it).

Let us do a few warmup exercises to familiarize ourselves with $\mathbb{E}$. We can construct $\mathbb{E}$ as follows. Set $\mathbb{E}_0 = \{0\}$, and for each $n > 0$ let $\mathbb{E}_n$ be the set of all complex numbers obtained either by applying a field operation to any pair of (not necessarily distinct) elements of $\mathbb{E}_{n-1}$ or by applying exp or log to any element of $\mathbb{E}_{n-1}$; of course, division by zero and taking the logarithm of zero are forbidden. Then it is clear that $\mathbb{E}$ is the union of all the $\mathbb{E}_n$. This shows in particular that $\mathbb{E}$ is countable, and that every element of $\mathbb{E}$ admits an explicit finite expression in terms of rational numbers, field operations, exp, and log.

Many familiar constants lie in $\mathbb{E}$, e.g.,

$$e = \exp(\exp(0)), \quad i = \exp\left(\frac{\log(-1)}{2}\right), \quad \text{and} \quad \pi = -i\log(-1).$$

Since $2\pi i \in \mathbb{E}$, we actually have access to all branches of the logarithm and not just the principal one, so all n of the nth roots of any $x \in \mathbb{E}$ are also in $\mathbb{E}$. It follows that all the roots of any polynomial equation with rational coefficients that is solvable in radicals lie in $\mathbb{E}$. Finally, formulas such as

$$x^{2/3} = \exp\left(\frac{2\log x}{3}\right), \quad \sin x = \frac{\exp(ix) - \exp(-ix)}{2i},$$

$$\tanh x = \frac{\exp(x) - \exp(-x)}{\exp(x) + \exp(-x)}, \quad \text{and} \quad \arccos x = -i\log\left(x + \exp\left(\frac{\log(x^2-1)}{2}\right)\right)$$

show that any expression involving "high-school" functions and elements of $\mathbb{E}$ is also in $\mathbb{E}$.

We hope that this brief discussion has persuaded the reader that $\mathbb{E}$ is the "right" precise definition of "the set of all complex numbers that can be written in closed form." Accepting this, we can reformulate Questions 1 and 2 as follows.

Conjecture 1. *The real root R of $x + e^x = 0$ is not in $\mathbb{E}$.*

Conjecture 2. *The roots r_1, r_2, r_3, r_4, and r_5 of $2x^5 - 10x + 5 = 0$ are not in $\mathbb{E}$.*

As far as I know, Conjecture 1 and Conjecture 2 are—perhaps surprisingly—still open. Thus we are still frustrated, but at least our frustration has been raised to a higher plane. The next section of this paper is devoted to partial results.

3. SCHANUEL'S CONJECTURE. Conjecture 1 is essentially due to Ritt, except that he asked the question with $\mathbb{L}$ instead of $\mathbb{E}$, since he was motivated by considerations different from ours. The best partial result I am aware of is due to Ferng-Ching Lin [**12**]. To state Lin's theorem, we must first recall *Schanuel's conjecture*.

Schanuel's Conjecture. *If $\alpha_1, \alpha_2, \dots, \alpha_n$ are complex numbers linearly independent over $\mathbb{Q}$, then the transcendence degree of the field $\mathbb{Q}(\alpha_1, e^{\alpha_1}, \alpha_2, e^{\alpha_2}, \dots, \alpha_n, e^{\alpha_n})$ over $\mathbb{Q}$ is at least n.*

Schanuel's conjecture implies many famous theorems and conjectures about transcendental numbers. For example, it implies the Lindemann-Weierstrass theorem: If $\alpha_1, \alpha_2, \dots, \alpha_n$ are algebraic numbers that are linearly independent over $\mathbb{Q}$, then $e^{\alpha_1}, e^{\alpha_2}, \dots, e^{\alpha_n}$ are algebraically independent over $\mathbb{Q}$ [**2**, Theorem 1.4]. Schanuel's conjecture also implies the Gelfond-Schneider theorem: If α_1 and α_2 are algebraic numbers for which there exist $\mathbb{Q}$-linearly independent numbers β_1 and β_2 such that $\alpha_1 = e^{\beta_1}$ and $\alpha_2 = e^{\beta_2}$, then β_1 and β_2 are linearly independent over the algebraic numbers. Baker's generalization [**2**, Theorem 2.1] of Gelfond-Schneider to an arbitrarily large finite number of α_i also follows from Schanuel's conjecture. It is an easy exercise (using $e^{\pi i} = -1$) to show that Schanuel's conjecture implies that e and π are algebraically independent, which is currently not known; it is not even known that $e + \pi$ is transcendental. A proof of Schanuel's conjecture would be big news, although at present it seems to be out of reach.

Let $\overline{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$. Then Lin's result is the following.

Theorem 1. *If Schanuel's conjecture is true and $f(x, y) \in \overline{\mathbb{Q}}[x, y]$ is an irreducible polynomial involving both x and y and $f(\alpha, \exp(\alpha)) = 0$ for some nonzero $\alpha \in \mathbb{C}$, then $\alpha \notin \mathbb{L}$.*

By taking $f(x) = x + y$ and noting that $\mathbb{E} \subseteq \mathbb{L}$, we see at once that Schanuel's conjecture implies Conjecture 1.

Conjecture 2 seems to be new. It is folklore that *general* polynomial equations (i.e., those with variable coefficients) cannot be solved in terms of the exponential and logarithmic functions, although nobody seems to have written down a complete proof; partial proofs may be found in [**9**, paragraph 513] and [**1**, p. 114]. The inexpressibility of an algebraic function in terms of exp and log does not, however, imply that *particular values* of an algebraic function cannot be expressed in terms of exp and log, just as some quintic equations with rational coefficients are solvable in radicals even though the general quintic is not.

The remainder of this section is devoted to proving the following result.

Theorem 2. *Schanuel's conjecture implies Conjecture* 1 *and Conjecture* 2.

As we just remarked, Lin has already shown that Schanuel's conjecture implies Conjecture 1, but we shall exploit the fact that Conjecture 1 is weaker than the conclusion of Theorem 1 to give a shorter proof. The proof we offer for Theorem 2 is joint work with Daniel Richardson. The reader may check that our arguments generalize readily to other transcendental equations such as $x = \cos x$. The reader may also verify that the only property of (1.2) we use is the unsolvability of its Galois group. We therefore obtain the following corollary of the proof.

Corollary 1. *If Schanuel's conjecture is true, then the algebraic numbers in $\mathbb{E}$ are precisely the roots of polynomial equations with integer coefficients that are solvable in radicals.*

Thus, Schanuel's conjecture implies that our notion of a "closed-form algebraic number" coincides with the usual one.

Although Conjecture 1 and Conjecture 2 involve quite different kinds of equations, it turns out that there is a single concept (a *reduced tower*) that is the key to both.

We need some preliminaries. If $A = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is a finite sequence of complex numbers, then for brevity we write A_i for the field $\mathbb{Q}(\alpha_1, e^{\alpha_1}, \alpha_2, e^{\alpha_2}, \dots, \alpha_i, e^{\alpha_i})$. In particular, $A_0 = \mathbb{Q}$.

Definition. A *tower* is a finite sequence $A = (\alpha_1, \alpha_2, \dots, \alpha_n)$ of nonzero complex numbers such that for all $i \in \{1, 2, \dots, n\}$, there exists some integer $m_i > 0$ such that $\alpha_i^{m_i} \in A_{i-1}$ or $e^{\alpha_i m_i} \in A_{i-1}$ (or both). A tower is *reduced* if the set $\{\alpha_i\}$ is linearly independent over $\mathbb{Q}$. If $\beta \in \mathbb{C}$, then a *tower for* β is a tower $A = (\alpha_1, \alpha_2, \dots, \alpha_n)$ such that $\beta \in A_n$.

For any $\gamma \in \mathbb{E}$, there exists a tower for γ. This is best explained by example. Suppose $\gamma = 4 + \log(1 + e^{(\log 2)/3})$. Then we may take

$$A = (\alpha_1, \alpha_2, \alpha_3) = \big(\log 2, (\log 2)/3, \log(1 + e^{(\log 2)/3})\big).$$

We can then take $m_i = 1$ for all i, because $e^{\alpha_1} = 2 \in A_0$, $\alpha_2 \in A_1$, and $e^{\alpha_3} \in A_2$. In general, we build up the expression for γ step by step, and if at step i we need to take the exponential of some number $\beta \in A_{i-1}$ we simply set $\alpha_i = \beta$; if we need to take the logarithm of some $\beta \in A_{i-1}$ then we set $\alpha_i = \log \beta$. With this construction, we never need to take $m_i > 1$, but the tower we obtain may not be reduced (as is the case in this example: $\alpha_1 - 3\alpha_2 = 0$). In order to be able to use Schanuel's conjecture, however, we need reduced towers, so our first goal is to show how to reduce a given tower.

Division Lemma. *Suppose $A = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is a tower and $q_1, q_2, \dots, q_n$ are nonzero integers. Then the sequence $B = (\beta_1, \beta_2, \dots, \beta_n)$ defined by $\beta_i = \alpha_i/q_i$ is also a tower, and $A_i \subseteq B_i$ for all i.*

Proof: Given any $i \in \{1, 2, \dots, n\}$, note that every $\gamma \in A_i$ is a rational function (with rational coefficients) of the numbers $\alpha_1, e^{\alpha_1}, \dots, \alpha_i, e^{\alpha_i}$. Now $\alpha_j = (\alpha_j/q_j)q_j = \beta_j q_j$ and $e^{\alpha_j} = e^{(\alpha_j/q_j)q_j} = (e^{\beta_j})^{q_j}$ for all j, so γ is also a rational function with rational coefficients of the numbers $\beta_1, e^{\beta_1}, \dots, \beta_i, e^{\beta_i}$, and hence $\gamma \in B_i$. So $A_i \subseteq B_i$ for all i.

Given any $i \in \{1, 2, \dots, n\}$, there is some integer $m_i > 0$ such that $\alpha_i^{m_i} \in A_{i-1}$ or $e^{\alpha_i m_i} \in A_{i-1}$. Consider first the case in which $\alpha_i^{m_i} \in A_{i-1}$. Then

$$\beta_i^{m_i} = \left(\frac{\alpha_i}{q_i}\right)^{m_i} \in A_{i-1} \subseteq B_{i-1}.$$

If on the other hand $e^{\alpha_i m_i} \in A_{i-1}$, then $e^{\beta_i(q_i m_i)} = e^{\alpha_i m_i} \in A_{i-1} \subseteq B_{i-1}$. Hence there is a positive integer m_i' (for example, $m_i' = q_i m_i$) such that $\beta_i^{m_i'} \in B_{i-1}$ or $e^{\beta_i m_i'} \in B_{i-1}$. So B is a tower. ∎

Reduction Lemma. *For any $\gamma \in \mathbb{E}$, there exists a reduced tower for γ.*

Proof: If $\gamma \in \mathbb{Q}$ then we may take A to be the empty sequence. Otherwise, suppose that every tower for γ is not reduced; we shall derive a contradiction. Choose such an A with n minimal; since $\gamma \notin \mathbb{Q}$, $n \geq 1$. Let i be the smallest integer such that $\{\alpha_1, \alpha_2, \dots, \alpha_i\}$ is linearly dependent. Then

$$\alpha_i = \sum_{j=1}^{i-1} \frac{p_j \alpha_j}{q_j} \tag{3.1}$$

for some integers $p_1, q_1, p_2, q_2, \dots, p_{i-1}, q_{i-1}$. We claim that the sequence

$$A' = \left(\frac{\alpha_1}{q_1}, \frac{\alpha_2}{q_2}, \dots, \frac{\alpha_{i-1}}{q_{i-1}}, \alpha_{i+1}, \alpha_{i+2}, \dots, \alpha_n \right)$$

is a tower for γ. Since A' is shorter than A, this contradicts the minimality of n and proves the theorem.

To prove the claim, note first that by the division lemma, the sequence

$$\left(\frac{\alpha_1}{q_1}, \frac{\alpha_2}{q_2}, \dots, \frac{\alpha_{i-1}}{q_{i-1}} \right)$$

is a tower. Next, note that (3.1) implies that $\alpha_i \in A'_{i-1}$ and also, by exponentiating, that e^{α_i} is a polynomial (in fact a monomial) in the numbers $e^{\alpha_1/q_1}, \dots, e^{\alpha_{i-1}/q_{i-1}}$, so that $e^{\alpha_i} \in A'_{i-1}$. By the division lemma, $A_{i-1} \subseteq A'_{i-1}$, so $A'_{i-1} \supseteq A_{i-1}(\alpha_i, e^{\alpha_i}) = A_i$. This ensures that the tower condition for A' is satisfied at the boundary between α_{i-1}/q_{i-1} and α_{i+1}, and also that $A'_{n-1} \supseteq A_n \ni \gamma$, proving the claim. ∎

Proof of Theorem 2. We first make a general remark. If $B = (\beta_1, \beta_2, \dots, \beta_n)$ is a reduced tower, then Schanuel's conjecture implies that for all i, *exactly one* of β_i and e^{β_i} is algebraic over B_{i-1}. For by the definition of a tower, *at least one* of the two is algebraic over B_{i-1}; this implies that the transcendence degree of B_i over $\mathbb{Q}$ is at most i for all i. Then because B is reduced, Schanuel's conjecture applies, so α_i and e^{α_i} cannot *both* be algebraic over B_{i-1}, and the transcendence degree of B_i over $\mathbb{Q}$ must be exactly i.

Now assume Schanuel's conjecture. We first prove Conjecture 1. Assume that $R \in \mathbb{E}$; we derive a contradiction. By the reduction lemma, there is a reduced tower $A = (\alpha_1, \alpha_2, \dots, \alpha_n)$ for R. Since e is transcendental, $R \notin \mathbb{Q}$ (for $R = p/q$ implies $e^p = (-p/q)^q$), so $n \geq 1$. By truncating the tower if necessary, we may assume that $R \notin A_i$ if $i < n$.

Let $A' = (\alpha_1, \alpha_2, \dots, \alpha_n, R)$. Then $R \in A'_n$, and the relation $R + e^R = 0$ shows that $e^R \in A'_n$ as well. By our "general remark," A' cannot be reduced. But A is reduced, so

$$R = \sum_{i=1}^{n} \frac{p_i \alpha_i}{q_i}$$

for some integers $p_1, q_1, p_2, q_2, \dots, p_n, q_n$. Moreover, $p_n \neq 0$ because $R \notin A_i$ for $i < n$. The relation $R + e^R = 0$ becomes

$$\sum_{i=1}^{n} \frac{p_i \alpha_i}{q_i} + \prod_{i=1}^{n} \left(e^{\alpha_i/q_i}\right)^{p_i} = 0. \tag{3.2}$$

Let $A'' = (\alpha_1/q_1, \alpha_2/q_2, \dots, \alpha_n/q_n)$. By the division lemma, A'' is a tower, and since A is reduced, A'' is reduced. But since $p_n \neq 0$, (3.2) shows that if α_n/q_n is algebraic over A''_{n-1} then so is e^{α_n/q_n}, and vice versa. By our "general remark," A'' cannot be reduced, and this gives our desired contradiction.

Now for Conjecture 2. We assume that the reader is familiar with the rudiments of Galois theory. Assume that $r_1 \in \mathbb{E}$; we derive a contradiction. By the reduction lemma there is a reduced tower $A = (\alpha_1, \alpha_2, \dots, \alpha_n)$ for r_1 (of course, unrelated to the "A" in the first part of this proof). For all i, let

$$\beta_i = \begin{cases} \alpha_i, & \text{if } \alpha_i \text{ is transcendental over } A_{i-1}; \\ e^{\alpha_i}, & \text{if } e^{\alpha_i} \text{ is transcendental over } A_{i-1}. \end{cases}$$

Then the β_i are algebraically independent and form a transcendence basis for A_n over $\mathbb{Q}$. Let $F = \mathbb{Q}(\beta_1, \beta_2, \dots, \beta_n)$. Clearly A_n is an extension by radicals of F. Let L be the Galois closure of A_n over F; then $\mathrm{Gal}(L/F)$ is solvable. If F' is the splitting field over $\mathbb{Q}$ of the polynomial (1.2), then $\mathrm{Gal}(F'/\mathbb{Q}) = S_5$ and $F' \cap F = \mathbb{Q}$ because $F'/\mathbb{Q}$ is an algebraic extension while $F/\mathbb{Q}$ is a purely transcendental extension. Therefore the compositum FF' is Galois over F with Galois group S_5 [**11**, Chapter 8, Theorem 1.12]. But $FF' \subseteq L$, so S_5 must be a homomorphic image of $\mathrm{Gal}(L/F)$. This is our desired contradiction, because every homomorphic image of a solvable group is solvable. ■

4. RELATED WORK AND OPEN PROBLEMS. Much of the work that has been done on fields such as $\mathbb{E}$, $\mathbb{L}$, or the field of elementary numbers has been motivated by problems in logic and computer algebra. A typical problem is this: given a complicated expression for a number in $\mathbb{E}$, how can you recognize if it equals zero? Clearly this is an important problem for designers of symbolic computation software. It is harder than it might seem at first glance, and is still not fully solved, although Richardson [**17**] has described an explicit procedure that takes a given elementary number and, if the procedure terminates, correctly says whether or not the number equals zero. He has also proved that if Schanuel's conjecture is true, then the procedure always terminates. This more or less solves the zero-recognition problem for elementary numbers (and *a fortiori* for $\mathbb{E}$ and $\mathbb{L}$) in practice.

The zero-recognition problem is closely related to a famous long-standing question of Tarski. Tarski proved that the first-order theory of the real numbers is decidable, which implies in particular that there is an algorithm for determining whether or not any given finite system of polynomial equations and inequalities has a solution in the reals [**8**, p. 340]. The proof proceeds by *quantifier elimination*, which we can think of roughly as follows: the statement that "there exists a solution" involves existential quantifiers, and quantifier elimination is a procedure for transforming such statements into ones that are quantifier-free. These are then easy to check because all that is involved is a zero-recognition problem for integers. After proving his theorem, Tarski asked if it could be extended to the first-order theory of the real numbers with exponentiation.

This problem is very hard, because it turns out that quantifier elimination is not possible in this theory. Moreover, checking quantifier-free statements involves the zero-recognition problem for expressions with exponentials, which is difficult. Great progress has been made recently, however. Macintyre [**13**] showed that if Schanuel's conjecture is true, then there is a decision procedure for the quantifier-

free statements. Then Wilkie proved in 1991 that the first-order theory of reals with exponentiation is model complete [**19**], which roughly means that quantifiers can "almost" be entirely eliminated. Building on this work, Macintyre and Wilkie showed that if Schanuel's conjecture is true, then the first-order theory of the real numbers with exponentiation is decidable [**14**]. In particular, from these methods one can extract a zero-recognition procedure for elementary numbers (again, contingent on Schanuel). See [**15**] for a splendid account of these and related results.

Zero-recognition in $\mathbb{E}$ should be easier than zero-recognition in the elementary numbers. Can one recognize zero in $\mathbb{E}$ without assuming Schanuel, or at least by assuming something weaker? The ideas of Macintyre [**13**] are a good starting-point here.

Another interesting open problem, posed by Thomas Colthurst (in a `sci.math` article posted on June 21, 1993), is to produce an explicit example of a number that is not in $\mathbb{E}$. Since $\mathbb{E}$ is countable, Cantor's diagonal argument gives us an algorithm for producing the decimal expansion of a non-EL number, but this is not very satisfying. It would be much nicer if we could prove, for example, that $\zeta(3) \notin \mathbb{E}$, but this seems difficult. Colthurst suggests that one might expect an expression of the form

$$F = \sum_{m=1}^{\infty} f(m)$$

to work, where $f(m)$ is a nonnegative function of m that approaches zero rapidly. For example, the sets $\mathbb{E}_n$ (defined in Section 2) are finite, so there exists some $\epsilon_n > 0$ such that any two distinct numbers in $\mathbb{E}_n$ differ in absolute value by at least ϵ_n. If one could find f with the property that, for all n,

$$\sum_{m=1}^{n} f(m) \in \mathbb{E}_n \quad \text{and} \quad \sum_{m=n+1}^{\infty} f(m) < \epsilon_n,$$

then F could not be in $\mathbb{E}_n$ for any n. This is probably too naïve, but perhaps something along these lines is feasible. Can f be chosen to be elementary?

Steve Finch asks for the relationship between EL numbers and holonomic constants. A *holonomic function* is a solution of a linear homogeneous ODE with polynomial coefficients. A *holonomic constant* is a value of a holonomic function at a rational regular point. *Singular holonomic constants* are values of holonomic functions in the vicinity of a singular point. Several famous constants such as π, $\zeta(3)$ and Catalan's constant are singular holonomic constants.

On the grounds that many high-school students are unfamiliar with complex numbers, one can ask for a "real analogue" of $\mathbb{E}$. What is the right definition? Such a real analogue would lack many of the nice properties of $\mathbb{E}$ (e.g., recall that if an irreducible cubic with rational coefficients has three distinct real roots, then they cannot be expressed using radicals alone if complex numbers are forbidden), but it might still be interesting.

Finally, we mention that Richardson (personal communication) has shown that if Schanuel's conjecture is false, then there is a counterexample involving only elementary numbers. Can this be strengthened to show that any counterexample must lie in $\mathbb{E}$?

We hope the reader is tempted to attack these relatively untouched questions.

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NOTES

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The Smallest Solution of $\phi(30n+1) < \phi(30n)$ Is ...

Greg Martin

In a previous issue of this MONTHLY, D. J. Newman [1] showed that for any positive integers a, b, c, and d with $ad \neq bc$, there exist infinitely many positive integers n for which $\phi(an+b) < \phi(cn+d)$, where $\phi(m)$ is the familiar Euler totient function, the number of positive integers less than and relatively prime to m. In particular, it must be the case that $\phi(30n+1) < \phi(30n)$ infinitely often; however, Newman mentions that there are no solutions of this inequality with $n \leq$ 20,000,000, and he states that a solution "is not explicitly available and it may be beyond the reach of any possible computers." The purpose of this note is to describe a method for computing solutions to inequalities of this type that avoids the need to factor large numbers. In particular, we explicitly compute the smallest number n satisfying $\phi(30n+1) < \phi(30n)$.

It is quite easy to compute values of n for which $\phi(30n+1)$ is relatively small by imposing many congruence conditions on n modulo primes, so that $30n+1$ is highly composite. However, the numbers n that arise in this way are quite large, having hundreds of digits. Computing $\phi(30n)$ exactly relies on the factorization of $30n$, which for integers of this size is not possible to find in a reasonable amount of time with today's computers and factoring algorithms. The idea underlying our method is to use partial knowledge of the factorization of a large number m to get an estimate for $\phi(m)$.

Claim 1. *Let p_i denote the i^{th} prime number. Let $q = \prod_{i=r+1}^{r+s} p_i$ for some positive integers r and s, and let m be an integer that is not divisible by any of the primes $p_1,\dots,p_r$. Then:*

(*a*) *if $m \leq q$, then m has at most s distinct prime factors*;

(b) *if m has at most s distinct prime factors, then $\phi(m)/m \geq \phi(q)/q$.*

Proof: Let t be the number of distinct prime factors of m, and let the prime factors be $p_{i_1},\dots,p_{i_t}$ with $i_1 < \cdots < i_t$. Since none of the primes $p_1,\dots,p_r$ divide m, it must be the case that $i_1 \geq r+1$, $i_2 \geq r+2$, and so on. If we define $k = \prod_{j=r+1}^{r+t} p_j$, we see that $k \leq \prod_{j=1}^{t} p_{i_j} \leq m$. But $m \leq q$ by assumption, and so $k \leq q$, which can be the case only if $t \leq s$. This proves part (a) of the claim.

For part (b), we use the fact that the function $\phi(m)/m$ can be written as a product over primes dividing m:

$$\frac{\phi(m)}{m} = \prod_{p|m}\left(1 - \frac{1}{p}\right).$$

With k defined as above, notice that

$$\frac{\phi(m)}{m} = \prod_{j=1}^{t}\left(1 - \frac{1}{p_{i_j}}\right) \geq \prod_{j=1}^{t}\left(1 - \frac{1}{p_{r+j}}\right) = \frac{\phi(k)}{k},$$

since $1 - 1/p$ is an increasing function of p. On the other hand, since $t \leq s$ by assumption, we have

$$\frac{\phi(k)}{k} = \prod_{j=r+1}^{r+t}\left(1 - \frac{1}{p_j}\right) \geq \prod_{j=r+1}^{r+s}\left(1 - \frac{1}{p_j}\right) = \frac{\phi(q)}{q},$$

since each $1 - 1/p$ is less than 1. This proves part (b) of the claim.

We now proceed to find the smallest solution of $\phi(30n + 1) < \phi(30n)$; our method applies to any inequality of the form $\phi(an + b) < \phi(cn + d)$. Clearly $30n + 1 \equiv 1 \pmod{30}$ for all n. Also, if n is a solution of $\phi(30n + 1) < \phi(30n)$, then

$$\frac{\phi(30n + 1)}{30n + 1} < \frac{\phi(30n)}{30n + 1} < \frac{\phi(30)n}{30n} = \frac{4}{15} = 0.26666\ldots,$$

since the inequality $\phi(ab) \leq \phi(a)b$ holds for all a and b. Thus it makes sense to look for numbers that satisfy both these conditions.

Claim 2. *Let $z = (p_4 p_5 \cdots p_{383})p_{385}p_{388}$. Then z is the smallest positive integer satisfying $z \equiv 1 \pmod{30}$ and $\phi(z)/z < 4/15$.*

Proof: A computation shows that z is indeed congruent to 1 (mod 30) and that

$$\frac{\phi(z)}{z} = \left(\prod_{i=4}^{383}\left(1 - \frac{1}{p_i}\right)\right)\left(1 - \frac{1}{p_{385}}\right)\left(1 - \frac{1}{p_{388}}\right) = 0.2666117\ldots < \frac{4}{15}.$$

Suppose m is an integer satisfying $m \equiv 1 \pmod{30}$ and $\phi(m)/m < 4/15$. Because of the congruence condition, m cannot be divisible by 2, 3, or 5. If we define $q_1 = \prod_{i=4}^{384} p_i$, then $\phi(q_1)/q_1 = 0.26671\ldots$, and so $\phi(q_1)/q_1 > \phi(m)/m$. Thus if we apply part (b) of Claim 1 with $r = 3$ and $s = 381$, we conclude that m must have more than 381 distinct prime factors.

Another computation reveals that the only numbers less than z that have at least 382 distinct prime factors are the numbers $p_4 p_5 \cdots p_{382} m'$, where $m' \in \{p_{383}p_{384}p_{385}, p_{383}p_{384}p_{386}, p_{383}p_{385}p_{386}, p_{383}p_{384}p_{387}, p_{383}p_{385}p_{387}, p_{384}p_{385}p_{386}, p_{383}p_{384}p_{388}, p_{383}p_{386}p_{387}\}$; and none of these numbers is congruent to 1 (mod 30).

Let us define $n = (z - 1)/30$, which by Claim 2 is both an integer and the smallest possible solution of $\phi(30n + 1) < \phi(30n)$. (Small wonder that we haven't stumbled across any solutions of this inequality—n has 1,116 digits!) It would be quite gracious of n to be an actual solution, and indeed it is.

First we show that $\phi(30n + 1)/(30n + 1) < \phi(30n)/30n$. We have already computed

$$\frac{\phi(30n + 1)}{30n + 1} = \frac{\phi(z)}{z} = 0.2666117\ldots. \tag{1}$$

It turns out that n is divisible by both 60 and $p_{4,874} = 47{,}279$, so define $n' = n/(60p_{4,874})$. We can compute that n' is not divisible by any of the first 80,000 primes. This computation can be done quickly by multiplying the primes together in blocks of 1,000, say, and computing the greatest common divisor of n' and the product. Since computing greatest common divisors is a very fast operation, checking that n' is not divisible by any of the first 80,000 primes takes only a few minutes on a workstation—much more reasonable than trying to factor a number with over a thousand digits.

Now define $q_2 = \prod_{i=80,001}^{80,186} p_i$. We compute that q_2 has 1,118 digits and so $q_2 > n > n'$. By using parts (a) and (b) of Claim 1 with $r = 80{,}000$ and $s = 186$, we see that $\phi(n')/n' \geq \phi(q_2)/q_2$. Therefore, since $\phi(ab) = \phi(a)\phi(b)$ when a and b are relatively prime, we compute

$$\frac{\phi(30n)}{30n} = \frac{\phi(30 \cdot 60p_{4,874})}{30 \cdot 60p_{4,874}} \frac{\phi(n')}{n'} \geq \frac{4}{15}\left(1 - \frac{1}{47{,}279}\right)\frac{\phi(q_2)}{q_2} = 0.2666124\ldots. \tag{2}$$

This shows that $\phi(30n + 1)/(30n + 1) < \phi(30n)/30n$, which doesn't quite imply that $\phi(30n + 1) < \phi(30n)$—only that $\phi(30n + 1) < \phi(30n)(1 + 1/(30n))$. However, the numbers computed in (1) and (2) differ in the sixth decimal place, while multiplying by $1 + 1/(30n)$ leaves a number unchanged until past the 1,100th decimal place.

Therefore the following theorem has been established.

Theorem. *The smallest solution of* $\phi(30n + 1) < \phi(30n)$ *is*

$n =$ 232, 909, 810, 175, 496, 793, 814, 049, 684, 205, 233, 780, 004, 859, 885, 966, 051, 235, 363, 345, 311, 075, 888, 344, 528, 723, 154, 527, 984, 260, 176, 895, 854, 182, 634, 802, 907, 109, 271, 610, 432, 287, 652, 976, 907, 467, 574, 362, 400, 134, 090, 318, 355, 962, 121, 476, 785, 712, 891, 544, 538, 210, 966, 704, 036, 990, 885, 292, 446, 155, 135, 679, 717, 565, 808, 063, 766, 383, 846, 220, 120, 606, 143, 826, 509, 433, 540, 250, 085, 111, 624, 970, 464, 541, 380, 934, 486, 375, 688, 208, 918, 750, 640, 674, 629, 942, 465, 499, 369, 036, 578, 640, 331, 759, 035, 979, 369, 302, 685, 371, 156, 272, 245, 466, 396, 227, 865, 621, 951, 101, 808, 240, 692, 259, 960, 203, 091, 330, 589, 296, 656, 888, 011, 791, 011, 416, 062, 631, 565, 320, 593, 772, 287, 118, 913, 728, 608, 997, 901, 791, 216, 356, 108, 665, 476, 306, 080, 740, 121, 528, 236, 888, 680, 120, 152, 479, 138, 327, 451, 088, 404, 280, 929, 048, 314, 912, 122, 784, 879, 758, 304, 016, 832, 436, 751, 532, 255, 185, 640, 249, 324, 065, 492, 491, 511, 072, 521, 585, 980, 547, 438, 748, 689, 307, 159, 363, 481, 233, 965, 802, 331, 725, 033, 663, 862, 618, 957, 168, 974, 043, 547, 448, 879, 663, 217, 971, 081, 445, 619, 618, 789, 985, 472, 074, 303, 100, 303, 636, 078, 827, 273, 695, 551, 162, 089, 725, 435, 110, 246, 701, 964, 021, 045, 849, 081, 811, 604, 427, 331, 227, 553, 783, 590, 821, 510, 091, 607, 567, 178, 842, 569, 576, 699, 548, 038, 217, 673, 171, 895, 383, 249, 326, 800, 667, 432, 993, 531, 186, 437, 659, 910, 632, 865, 419, 892, 370, 957, 722, 154, 266, 351, 039, 808, 548, 150, 828, 868, 968, 820, 675, 198, 820, 381, 135, 523, 646, 361, 202, 383, 915, 218, 571, 017, 801, 463, 011, 491, 108, 784, 343, 253, 284, 393, 511, 650, 254, 506, 597, 923, 969, 653, 616, 813, 897, 710, 621, 756, 693, 827, 471, 154, 701, 151, 222, 320, 443, 347, 408, 180, 047, 964, 860.

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A Matrix Representation for Euler's Constant, γ

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Euler's constant, $\gamma = 0.5772156649\ldots$ can be represented as the product of an infinite row vector, the inverse of a $\mathbb{Z}^+ \times \mathbb{Z}^+$ lower triangular matrix, and an infinite $\mathbb{Z}^+ \times \mathbf{1}$ column vector, all with entries that are either zero or simple unit fractions.

Observe that for $\mathbb{Z}^+ \times \mathbb{Z}^+$ lower triangular matrices, the end result of all arithmetic matrix operations, matrix inversion, application to infinite column vectors, etc., has appropriate n-th truncation equal to that obtained by first truncating all matrices, and then carrying out the operations. Hence these operations may be performed and the familiar identities of linear algebra continue to hold in this context.

Theorem. *Let* $\mathbf{u}$ *be the row vector* $\{u_k = 1/k : k \in \mathbb{Z}^+\}$, *where* $\mathbb{Z}^+$ *denotes the positive integers, and let* $\mathbf{M}$ *be the matrix with entries* $\{m_{ij} = 1/(i-j+1)$ if $j \le i$, $m_{ij} = 0$ if $j > i : i, j \in \mathbb{Z}^+\}$. *Let* $\mathbf{v}$ *be the column vector* $\{v_n = 1/(n+1) : n \in \mathbb{Z}^+\}$. *Then the product* $\mathbf{u}(\mathbf{M}^{-1}\mathbf{v})$ *exists (as a convergent series), and is equal to Euler's constant,*

$$\gamma = \lim_{m \to \infty} \left(\sum_{n=1}^{m} \frac{1}{n} - \ln(m) \right) \approx 0.5772156649\ldots.$$

Explicitly,

$$\gamma = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{2} & 1 & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 1 & 0 & \cdots \\ \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \\ \frac{1}{5} \\ \frac{1}{6} \\ \vdots \end{bmatrix} \right)$$

Proof: Substituting $t = 1 - e^{-x}$ in the standard definite integral

$$\gamma = \int_o^\infty \left(\frac{1}{1 - e^{-x}} - \frac{1}{x} \right) e^{-x}\, dx \quad \text{yields} \quad \gamma = \int_0^1 \left(1 + \frac{t}{\ln(1-t)} \right) \frac{dt}{t}.$$

Write

$$\frac{t}{\ln(1-t)} = -\sum_{k=0}^{\infty} c_k t^k,$$

where the coefficients c_k are obtained from the formal division of power series. Since

$$\frac{\ln(1-t)}{t} = -\sum_{k=1}^{\infty} \frac{t^{k-1}}{k} \quad \text{on} \quad (-1, +1),$$

$\sum_{k=0}^{\infty} c_k t^k$ converges on some interval around 0, and $1 = (\sum_{k=1}^{\infty} t^{k-1}/k)(\sum_{k=0}^{\infty} c_k t^k)$ on this interval. Since $c_0 = \lim_{t \to 0} - t/\ln(1-t) = 1$, this is equivalent to the system of linear equations

$$\begin{aligned}
-\frac{1}{2} &= c_1 \\
-\frac{1}{3} &= \frac{c_1}{2} + c_2 \\
-\frac{1}{4} &= \frac{c_1}{3} + \frac{c_2}{2} + c_3 \\
-\frac{1}{5} &= \frac{c_1}{4} + \frac{c_2}{3} + \frac{c_3}{2} + c_4 \\
&\vdots
\end{aligned}$$

This system has the matrix form:

$$\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{3} \\ -\frac{1}{4} \\ -\frac{1}{5} \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & 1 & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{2} & 1 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ \vdots \end{bmatrix}$$

Thus we have $-\mathbf{v} = \mathbf{Mc}$, where $\mathbf{c}$ denotes the column vector $\{c_k : k \in \mathbb{Z}^+\}$. Eliminating the constant terms between two successive equations where c_{k-1} and c_k have unit coefficients, we have

$$c_k = \frac{1}{k+1} \sum_{j=1}^{k-1} \frac{j}{(k-j)(k-j+1)} c_j .$$

Consequently, by induction, $c_1 < 0, c_2 < 0, \cdots, c_{k-1} < 0$ implies $c_k < 0$, and again by induction $-1/(k+1) < c_k < 0$ $(k > 1)$. The latter inequality assures the convergence of $\sum_{k=1}^{\infty} c_k/k$. Therefore, using Abel's Theorem, we have

$$\gamma = -\lim_{x \to 1-} \int_0^x \left(\sum_{k=1}^{\infty} c_k t^k \right) \frac{dt}{t} = -\lim_{x \to 1-} \sum_{k=1}^{\infty} \frac{c_k}{k} x^k = -\sum_{k=1}^{\infty} \frac{c_k}{k} = -\mathbf{uc}.$$

Then $\mathbf{c} = (\mathbf{M}^{-1}\mathbf{M})\mathbf{c} = \mathbf{M}^{-1}(\mathbf{Mc}) = -\mathbf{M}^{-1}\mathbf{v}$. Using $\gamma = -\mathbf{uc}$, we obtain the result $\gamma = \mathbf{u}(\mathbf{M}^{-1}\mathbf{v})$.

We note that $\mathbf{M}$ is an unbounded operator on l_2. For example, using the Euclidean norm the sequence of unit vectors defined by

$$\mathbf{w}_m = \left\{ \{\mathbf{w}_m\}_n = \frac{1}{\sqrt{m}} \quad \text{for } n \le m, \quad \text{and } = 0 \text{ for } n > m \right\}$$

transforms into the sequence $\mathbf{Mw}_m$, which diverges because

$$\|\mathbf{Mw}_m\|^2 = \frac{1}{m} \sum_{n=1}^{m} \left(\sum_{k=1}^{n} \frac{1}{k} \right)^2$$

grows faster than $\ln(m!)/m \approx \ln m$, as m increases.

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More on a Mean Value Theorem Converse

H. Fejzić and D. Rinne

In a recent MONTHLY article Tong and Braza considered two possible versions of a converse to the Mean Value Theorem [**2**]. For $c \in (a, b)$, a continuous function f on $[a, b]$ that is differentiable on (a, b) satisfies the

1. Weak Form at c if $f'(c) = \frac{f(\beta) - f(\alpha)}{\beta - \alpha}$ for some interval $(\alpha, \beta) \subset (a, b)$, and the
2. Strong Form at c if $f'(c) = \frac{f(\beta) - f(\alpha)}{\beta - \alpha}$ for some interval $(\alpha, \beta) \subset (a, b)$ with $c \in (\alpha, \beta)$.

In [**2**] the authors give a function that fails the Weak Form (and so fails both forms) at all values in a countable closed set. Borwein and Wang provided a function that fails the Weak Form on a residual set (one whose complement is of first category) that is of Lebesgue measure zero [**1**].

We show that a differentiable function can fail the Weak Form on a set that is both residual and of relative measure arbitrarily close to 1 while the Strong Form must hold on some subset of positive Lebesgue measure. In the rest of this Note *measure* means Lebesgue measure, denoted by λ.

We consider $[a, b] = [0,1]$ and let Z be any measurable set in $[0, 1]$ with $\lambda(Z) < 1$. Let $E \subset [0, 1] \setminus Z$ be an F_σ set with $\lambda(E) = \lambda([0, 1] \setminus Z) > 0$ and E having density 1 at each $x \in E(\lim_{\epsilon \to 0} \lambda(E \cap (x - \epsilon, x + \epsilon))(2\epsilon)^{-1} = 1)$. Let g be an approximately continuous function (at each x the restriction of g to some subset with density 1 at x is continuous at x) such that:

$$\begin{aligned} &1.\ 0 < g(x) \le 1 \quad \text{for } x \in E, \quad \text{and} \\ &2.\ g(x) = 0 \quad \text{for } x \notin E. \end{aligned} \tag{1}$$

A construction of such functions can be found in Zahorski [**3**]. Since g is bounded

and approximately continuous it is the derivative of its integral $f(x) = \int_0^x g(t)\,dt$. Therefore $f' \equiv 0$ on Z. We can pick Z to be dense in $[0,1]$ and of measure arbitrarily close to 1 with E having positive measure in every subinterval of $[0,1]$. Then f is strictly increasing and thus has no difference quotient equal to zero. Hence f fails the Weak Form at every point of $\{x | f'(x) = 0\}$ and thus at every point of Z. Since $\{x | f'(x) = 0\}$ is a dense G_δ (it's the complement of the F_σ set E), f fails the Weak Form on a residual set.

However, the following theorem shows that a differentiable function cannot fail the Weak Form almost everywhere.

Theorem 1. *If f is a continuous function on $[a, b]$ that is differentiable on (a, b), then f satisfies the Strong Form on a subset of $[a, b]$ that has positive measure in every subinterval.*

Proof: Let $[\alpha, \beta] \subset [a, b]$. We may assume that f is not linear on any subinterval of $[\alpha, \beta]$ since it would then obviously satisfy the Strong Form there. Let

$$h(x) = \begin{cases} \dfrac{f(x) - f(\alpha)}{x - \alpha} & \text{for } \alpha < x \leq \beta \\ f'(\alpha) & \text{for } x = \alpha \end{cases}$$

Then h is continuous on $[\alpha, \beta]$ and $h([\alpha, \beta])$ is some nondegenerate interval $[r, s]$. Since h can have only countably many local extrema we can pick $u \in (\alpha, \beta)$ so that $h(u)$ is not a local extremum. Let c be a point in (α, u) with $f'(c) = h(u)$. Using $p = (c + u)/2$ we see that $f'(c)$ is in the interior of $h([p, \beta])$. Call this interior I. Let g be the restriction of f to the interval $[\alpha, p]$. Then $G = (g')^{-1}(I) \neq \phi$ since it contains c and thus $\lambda(G) > 0$ by the Denjoy-Clarkson Property (the inverse image under a derivative of an open interval is either empty or of positive measure). For each $x \in G$, there is a $y \in [p, \beta]$ with $f'(x) = g'(x) = h(y) = (f(y) - f(\alpha))/(y - \alpha)$. Since $\alpha < x < y$, f satisfies the Strong Form at x. ■

As a final comment, we point out that a differentiable function can fail the Strong Form on a set of positive measure and still satisfy the Weak Form on all of (a, b). As an example we can simply extend our function g in (1) to the interval $[0, 4]$ as follows: Let

$$G(x) = \begin{cases} g(x) & 0 \leq x \leq 1 \\ -g(1)(x - 2) & 1 < x \leq 2 \\ 0 & 2 < x \leq 3 \\ (x - 3) & 3 < x \leq 4 \end{cases}$$

and set $F(x) = \int_0^x G(t)\,dt$. Then F still fails the Strong Form on the set Z above but satisfies the Weak Form on $(0, 4)$. This is because $0 \leq G = F' < 1$ on $(0, 4)$ while the difference quotients for F inside the interval $(2, 4)$ assume all values in $[0, 1)$.

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An Elegant Continued Fraction for π

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The regular continued fraction for π begins as follows [3, p. 23]:

$$\pi = 3 + \frac{1}{7} + \frac{1}{15} + \frac{1}{1} + \frac{1}{292} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{1}{3} + \frac{1}{1} + \frac{1}{14} + \cdots. \tag{1}$$

There is no known regularity to the partial denominators in (1) and the only known means to obtain them is to compute them one-by-one from a known decimal approximation for π. Lord Brouncker (1620–1686), the first president of the Royal Society of London, gave (without proof around 1659) the first recorded infinite continued fraction [3, p. 2]:

$$\frac{4}{\pi} = 1 + \frac{1^2}{2} + \frac{3^2}{2} + \frac{5^2}{2} + \frac{7^2}{2} + \frac{9^2}{2} + \frac{11^2}{2} + \frac{13^2}{2} + \cdots. \tag{2}$$

In 1775, according to [1, p. 131], Euler gave a proof of the validity of (2) by showing that

$$\arctan x = \frac{x}{1} + \frac{1^2x^2}{3 - x^2} + \frac{3^2x^2}{5 - 3x^2} + \frac{5^2x^2}{7 - 5x^2} + \cdots \tag{3}$$

is equivalent to the power series representation

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad -1 \le x \le 1.$$

Brouncker's result can be obtained by setting $x = 1$ in (3).

The following continued fraction expansion for the principal branch of the analytic function $\arctan z$, valid for all z in the complex plane not on the imaginary axis from i to $+i\infty$ and from $-i$ to $-i\infty$, is well known [3, p. 202]:

$$\arctan z = \frac{z}{1} + \frac{1^2z^2}{3} + \frac{2^2z^2}{5} + \frac{3^2z^2}{7} + \frac{4^2z^2}{9} + \cdots. \tag{4}$$

Setting $z = 1$ in (4) leads to

$$\frac{\pi}{4} = \frac{1}{1} + \frac{1^2}{3} + \frac{2^2}{5} + \frac{3^2}{7} + \frac{4^2}{9} + \cdots. \tag{5}$$

Although they are not formulas for π itself, the classical continued fractions (2) and (5) are attractive because of the simple expressions for all of their partial numerators and denominators. Our contribution is the following continued fraction for π itself, whose partial numerators and denominators are easily described and remembered. Though the tools to derive it have long been available, to our knowledge, this formula has not yet appeared in the literature.

Theorem 1.

$$\pi = 3 + \frac{1^2}{6} + \frac{3^2}{6} + \frac{5^2}{6} + \frac{7^2}{6} + \frac{9^2}{6} + \frac{11^2}{6} + \frac{13^2}{6} + \cdots. \tag{6}$$

Proof: We think it is of interest to show in several different ways that (6) is valid. Perron [**5**, p. 35] gives the following representation, which he attributes to Stieltjes:

$$x + \frac{1^2 - n^2}{2x} \, + \, \frac{3^2 - n^2}{2x} \, + \, \frac{5^2 - n^2}{2x} \, + \cdots = 4 \cdot \frac{\Gamma\left(\dfrac{x+3+n}{4}\right)\Gamma\left(\dfrac{x+3-n}{4}\right)}{\Gamma\left(\dfrac{x+1+n}{4}\right)\Gamma\left(\dfrac{x+1-n}{4}\right)}, \tag{7}$$

where $x > 0$ and $1 > n^2 > -\infty$. Setting $n = 0$ in (7) gives

$$x + \frac{1^2}{2x} + \frac{3^2}{2x} + \frac{5^2}{2x} + \cdots = 4 \cdot \frac{\Gamma\left(\dfrac{x+3}{4}\right)\Gamma\left(\dfrac{x+3}{4}\right)}{\Gamma\left(\dfrac{x+1}{4}\right)\Gamma\left(\dfrac{x+1}{4}\right)}, \tag{8}$$

which is a formula also obtained by Ramanujan and Preece according to Perron [**5**, p. 36]. To obtain (6) we have only to substitute $x = 3$ in (8) and employ the properties $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(1) = 1$, and $\Gamma(x+1) = x\Gamma(x)$ of the Γ-function. It is surprising that apparently Ramanujan either was not aware of, or else did not choose to record this result. To show how we really arrived at (6) the first time, we need the following result [**5**, Satz 1.13, p. 28] relating to what are known as *Bauer-Muir transformations* of continued fractions; see [**4**].

Theorem 2. (*a*) *If both continued fractions*

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots \quad \textit{and}$$

$$b_0 + r_0 + \frac{\varphi_1}{b_1 + r_1} + \frac{a_1 \dfrac{\varphi_2}{\varphi_1}}{b_2 + r_2 - r_0 \dfrac{\varphi_2}{\varphi_1}} + \frac{a_2 \dfrac{\varphi_3}{\varphi_2}}{b_3 + r_3 - r_1 \dfrac{\varphi_3}{\varphi_2}} + \cdots,$$

where $\varphi_\nu = a_\nu - r_{\nu-1}(b_\nu + r_\nu)$, *have positive elements and if both converge, then they have the same value.* (*b*) *If the first continued fraction has positive elements and it converges and if* $r_\nu \geq 0$ *from a certain* ν *on, then the second continued fraction also converges and it has the same value as the first.*

The second continued fraction in Theorem 2 is called the *Bauer-Muir transform* of the first one. On page 35 of [**5**] is the expansion

$$z \cot \frac{\pi z}{4} = 1 + \frac{1^2 - z^2}{2} + \frac{3^2 - z^2}{2} + \frac{5^2 - z^2}{2} + \frac{7^2 - z^2}{2} + \cdots, \tag{9}$$

which is valid for all complex z. If we apply Theorem 2 to this continued fraction with $z = x \in (-1, 1)$ and

$$a_n = (2n-1)^2 - x^2, \quad b_n = 2, \quad r_n = 2n - 1, \quad \varphi_n = 4 - x^2,$$

we obtain

$$x \cot \frac{\pi x}{4} = \frac{2^2 - x^2}{3} + \frac{1^2 - x^2}{6} + \frac{3^2 - x^2}{6} + \frac{5^2 - x^2}{6} + \frac{7^2 - x^2}{6} + \cdots. \tag{10}$$

Taking the limit of both sides of (9) as $z \to 0$ gives Brounker's result (2), and taking the limit of both sides of (10) as $x \to 0$ leads to (6) upon taking reciprocals.

It would be nice if the speed of convergence of (6) was in accordance with its beauty, but unfortunately this is not the case. In support of this slowness assertion the 100th approximant of (6) rounds to 3.14159241, whereas both π and the 4th approximant of its regular continued fraction expansion (1) round to 3.14159265. If the expansions (2) and (5) are used to approximate π, the 11th approximant of (5) gives 3.14159265 as an approximation, but Brouncker's continued fraction (2) converges so slowly that its 1000th approximant leads to the poor estimate of 3.14259165 for π. As another source of information about π, we recommend to the reader the recent book [**2**].

Addendum:. The formula (6) was used as a logo for the conference on continued fractions that was held at the University of Missouri-Columbia in late May 1998. At this conference D. Bowman of the University of Illinois mentioned in a personal conversation that he had another approach to deriving (6). Bowman starts with the result

$$\frac{\pi - 3}{4} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k(2k+1)(2k+2)} = \frac{1}{4} \sum_{k=1}^{\infty} (-1)^{k-1} \left(\frac{1}{k} + \frac{1}{k+1} - \frac{4}{2k+1} \right) \tag{11}$$

and then makes use of the fact that for $a_k \neq 0$ the series $\sum_{k=1}^{\infty} (-1)^{k-1}/a_k$ and the continued fraction

$$\frac{1}{a_1} + \frac{a_1^2}{a_2 - a_1} + \frac{a_2^2}{a_3 - a_2} + \frac{a_3^2}{a_4 - a_3} + \cdots \tag{12}$$

are equivalent, that is, the nth partial sum of the series and the nth approximant of the continued fraction are equal. This connection between series and continued fractions can be derived easily from a result of Euler (see [**5**, p. 17] or [**3**, p. 37]), or it can be proved directly by induction. After replacing a_k by $2k(2k+1)(2k+2)$ in (12) and calculating $a_{k+1} - a_k = 24(k+1)^2$, we are led to the representation (6) through a simple cancellation process that preserves the equivalence of the continued fractions involved. Bowman mentioned that his approach to verifying (6) gives as a welcome by-product some immediate truncation error information. Because of the series-continued fraction equivalence and the alternating nature of the first series in (11), we have $|\pi - f_n| \leq 1/((n+1)(n+2)(2n+3))$, where f_n is the nth approximant of (6).

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The Reciprocity Law for Dedekind Sums via the Constant Ehrhart Coefficient

Matthias Beck

1. Introduction. The Dedekind sum can be defined for two relatively prime positive integers a, b by

$$\mathfrak{s}(a,b) = \frac{1}{4b}\sum_{k=1}^{b-1} \cot\frac{\pi k a}{b} \cot\frac{\pi k}{b}.$$

These sums appear in various branches of mathematics: number theory, algebraic geometry, and topology; they have consequently been studied extensively in various contexts. These include the quadratic reciprocity law [13], random number generators [12], group actions on complex manifolds [9], and lattice point problems ([14] or [5]). Dedekind was the first to show the following reciprocity law [3]:

$$\mathfrak{s}(a,b) + \mathfrak{s}(b,a) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a}\right) \tag{1}$$

He was led naturally to this reciprocity law by considering the η-function $\eta(\tau) = e^{\pi i \tau/12}\prod_{m=1}^{\infty}(1 - e^{2\pi i m \tau})$ on the complex upper half plane and transforming it under the action of the modular group $SL_2(\mathbb{Z})$.

Gauß's law of quadratic reciprocity, for example, follows easily from (1); see [13] or [16]. We note that $\mathfrak{s}(a,b) = \mathfrak{s}(a \bmod b, b)$. Combining this with the reciprocity law (1), one obtains a polynomial-time algorithm for computing $\mathfrak{s}(a,b)$, similar in spirit to the Euclidean algorithm. From this point of view, it is not surprising (though not obvious) that $\mathfrak{s}(a,b)$ can be expressed efficiently in terms of the continued fraction expansion of a/b; see [8] or [19].

Rademacher was one of the pioneers in the use of Dedekind sums [17]; in fact, he found several proofs of (1) [16]. We present yet another proof, which establishes a simple connection with lattice point enumeration in polytopes. The reciprocity law (1) follows readily once the reader is familiar with the computation of the coefficients of the Ehrhart polynomial for a lattice polytope.

2. COUNTING LATTICE POINTS. Let $\mathbb{Z}^n \subset \mathbb{R}^n$ be the n-dimensional integer lattice, and let $\mathscr{P}$ be an n-dimensional lattice polytope in $\mathbb{R}^n$, so $\mathscr{P}$ is a compact simplicial complex of pure dimension n whose vertices lie on the lattice. For $t \in \mathbb{N}$, denote by $L(\mathscr{P}, t)$ the number of lattice points in the closure of the dilated polytope $t\mathscr{P} := \{tx : x \in \mathscr{P}\}$. Ehrhart proved that $L(\mathscr{P}, t)$ is a polynomial in t of degree n [6]. Moreover,

$$L(\mathscr{P}, t) = \mathrm{Vol}(\mathscr{P})t^n + \tfrac{1}{2}\mathrm{Vol}(\partial\mathscr{P})t^{n-1} + \cdots + \chi(\mathscr{P}).$$

Here, $\mathrm{Vol}(\partial\mathscr{P})$ denotes the surface area of $\mathscr{P}$ normalized with respect to the sublattice on each face of $\mathscr{P}$, and $\chi(\mathscr{P})$ is the Euler characteristic of $\mathscr{P}$. We note that, for convex polytopes $\mathscr{P}$, $\chi(\mathscr{P}) = 1$ [6].

In this paper, we focus on the case $\mathbb{R}^2$, where Ehrhart's result is known as Pick's Theorem; see [7] or [4]: For a convex lattice polytope $\mathscr{P} \in \mathbb{R}^2$,

$$L(\mathscr{P}, t) = At^2 + \tfrac{1}{2}Bt + 1,$$

where A is the area and B is the number of boundary lattice points of $\mathscr{P}$.

In the general case, the other coefficients of $L(\mathscr{P}, t)$ are not as easily accessible. In fact, until quite recently a method of computing these coefficients was unknown. There has been recent progress in this direction ([**1**], [**2**], [**10**], and [**11**]); Diaz and Robins found a way of proving a cotangent representation for the generating function $\Sigma_{t=0}^{\infty} L(\mathscr{P}, t)e^{-2\pi st}$, thereby deriving a formula for the Ehrhart coefficients of $L(\mathscr{P}, t)$ [**5**]. For our purposes, the following result (a straightforward consequence of [**5**, Corollary 1]) is sufficient:

Theorem. *Let $\mathscr{P}$ denote the simplex in $\mathbb{R}^n$ with the vertices $(0, \dots, 0)$, $(a_1, 0, \dots, 0)$, $(0, a_2, 0, \dots, 0), \dots, (0, \dots, 0, a_n)$, where $a_1, \dots, a_n \in \mathbb{N}$ are pairwise coprime. Denote the corresponding Ehrhart polynomial by $L(\mathscr{P}, t) = \Sigma_{j=0}^{n} c_j t^j$. Then c_m is the coefficient of $s^{-(m+1)}$ in the Laurent expansion at $s = 0$ of*

$$\frac{\pi^{m+1}}{m!\,2^{n-m}p} \sum_{r=1}^{p} \left(1 + \coth\frac{\pi}{a_1}(s + ir)\right)\left(1 + \coth\frac{\pi}{a_2}(s + ir)\right) \cdots \left(1 + \coth\frac{\pi}{a_n}(s + ir)\right)\left(1 + \coth\frac{\pi}{p}(s + ir)\right),$$

where $p = a_1 \cdots a_n$.

The appearance of cotangent products in this result leads us to expect Dedekind sums in some form within the coefficients of the Ehrhart polynomial, thus also within the formulas for the number of lattice points in simplices. In fact, the nontrivial cases of dimension three [**15**] and four [**18**] involve classical Dedekind sums. Both formulas can be obtained easily through the Theorem.

We use this result in an indirect way. Precisely, we compute c_0 according to the Theorem, and make use of the fact that $c_0 = \chi(\mathscr{P}) = 1$. Dedekind's reciprocity law (1) follows from this idea if we consider the case of dimension $n = 2$.

3. PROOF OF THE RECIPROCITY LAW. According to the Theorem, for coprime a and b we have to find the coefficient of s^{-1} of the Laurent series at $s = 0$ of

$$\frac{\pi}{4ab} \sum_{r=1}^{ab} \left(1 + \coth\frac{\pi}{a}(s + ir)\right)\left(1 + \coth\frac{\pi}{b}(s + ir)\right)\left(1 + \coth\frac{\pi}{ab}(s + ir)\right). \quad (2)$$

The Laurent expansion of each factor depends on r:

$$1 + \coth\frac{\pi}{c}(s + ir) = \begin{cases} S_c := \dfrac{c}{\pi}s^{-1} + 1 + \dfrac{\pi}{3c}s + O(s^3) & \text{if } c \mid r \\[2ex] R_c := 1 + \coth\dfrac{\pi i r}{c} + O(s) & \text{if } c \nmid r \end{cases}$$

To keep track of the various cases, we introduce the notation

$$\chi_c = \begin{cases} 1 \text{ if } c \mid r \\ 0 \text{ if } c \nmid r, \end{cases}$$

so that we can write $1 + \coth \pi(s + ir)/c = S_c \chi_c + R_c(1 - \chi_c)$, and (2) becomes

$$\sum_{r=1}^{ab} (S_a \chi_a + R_a(1 - \chi_a))(S_b \chi_b + R_b(1 - \chi_b))(S_{ab} \chi_{ab} + R_{ab}(1 - \chi_{ab})).$$

Now, expand this into all 8 terms, and consider each summand according to the number of S_c factors:

1. Terms with one S_c factor are

$$\begin{aligned} S_a \chi_a R_b(1 - \chi_b) R_{ab}(1 - \chi_{ab}) &= S_a R_b R_{ab} \chi_a(1 - \chi_b - \chi_{ab} + \chi_{ab}) \\ &= S_a R_b R_{ab}(\chi_a - \chi_{ab}) \end{aligned} \tag{3}$$

and, similarly,

$$R_a(1 - \chi_a) S_b \chi_b R_{ab}(1 - \chi_{ab}) = R_a S_b R_{ab}(\chi_b - \chi_{ab}). \tag{4}$$

The summand with S_{ab} is zero (note that $\chi_a \chi_{ab} = \chi_b \chi_{ab} = \chi_{ab}$, and $\chi_a \chi_b = \chi_{ab}$). To compute the contribution of (3), note that the support of $\chi_a - \chi_{ab}$ in $\{1, \ldots, ab\}$ is $\{ka : 1 \le k \le b - 1\}$; thus its contribution to (2) is

$$\begin{aligned} &\frac{\pi}{4ab} \cdot \frac{a}{\pi} \sum_{k=1}^{b-1} \left(1 + \coth \frac{\pi ika}{b}\right)\left(1 + \coth \frac{\pi ika}{ab}\right) \\ &= \frac{1}{4b} \sum_{k=1}^{b-1} \left(1 - i \cot \frac{\pi ka}{b}\right)\left(1 - i \cot \frac{\pi k}{b}\right) \\ &= \frac{1}{4b} \sum_{k=1}^{b-1} 1 - \cot \frac{\pi ka}{b} \cot \frac{\pi k}{b} + i \cdots = \frac{1}{4} - \frac{1}{4b} - \mathfrak{s}(a, b). \end{aligned}$$

The imaginary part in the preceding sum has to be zero, because the original generating function is real. Similarly, (4) gives a contribution of $\frac{1}{4} - \frac{1}{4}a^{-1} - \mathfrak{s}(b, a)$.

2. There are no terms with two S_c factors, because

$$S_a \chi_a S_b \chi_b R_{ab}(1 - \chi_{ab}) = S_a S_b R_{ab} \chi_{ab}(1 - \chi_{ab}) = 0$$

and

$$S_a \chi_a R_b(1 - \chi_b) S_{ab} \chi_{ab} = S_a R_b S_{ab} \chi_{ab}(1 - \chi_b) = 0.$$

3. Finally, the term $S_a \chi_a S_b \chi_b S_{ab} \chi_{ab} = S_a S_b S_{ab} \chi_{ab}$ has support $\{ab\}$, and gives a contribution of

$$\begin{aligned} &\frac{\pi}{4ab}\left(\frac{a}{\pi}\frac{b}{\pi}\frac{\pi}{3ab} + \frac{a}{\pi}\frac{ab}{\pi}\frac{\pi}{3b} + \frac{b}{\pi}\frac{ab}{\pi}\frac{\pi}{3a} + \frac{a}{\pi} + \frac{b}{\pi} + \frac{ab}{\pi}\right) \\ &= \frac{1}{12}\left(\frac{1}{ab} + \frac{a}{b} + \frac{b}{a}\right) + \frac{1}{4}\left(\frac{1}{b} + \frac{1}{a} + 1\right). \end{aligned}$$

Adding all contributions, we arrive at

$$1 = c_0 = \frac{3}{4} + \frac{1}{12}\left(\frac{1}{ab} + \frac{a}{b} + \frac{b}{a}\right) - \mathfrak{s}(a, b) - \mathfrak{s}(b, a),$$

the desired reciprocity law (1).

The same method applied to dimension $n = 3$ does not give any further results. However, for $n = 4$, higher dimensional Dedekind sums **[20]** appear within the computations, so that this case is likely to provide new results.

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Author's comment: In the course of proofreading, it was discovered that Pommersheim made an observation in his paper **[14]** similar to our idea of equating the Euler characteristic with the given cotangent Laurent expansion. His approach used toric varieties but translates into an equivalent statement.

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THE EVOLUTION OF...

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Riemann's Dissertation and Its Effect on the Evolution of Mathematics

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Translated from the German by Abe Shenitzer†

A short account of the contents of the dissertation. Riemann's doctoral dissertation of 1851 is titled *Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse* (*Foundations for a general theory of functions of a variable complex quantity*) [**1**, 3–43]. It is of modest size. In discussing it we use modern terms.

Riemann defines holomorphic functions as complex single-valued functions on Riemann surfaces satisfying the Cauchy-Riemann differential equations. Riemann also worked with functions that were holomorphic except for finite poles in $\mathbb{C}$. Such meromorphic functions are viewed as conformal mappings between two Riemann surfaces. We must always think of the complex plane as extended by the addition of the point ∞ (as the Riemann complex number sphere or as a complex projective straight line).

Functions must be thought of not as given by expressions but as *determined* (to within arbitrary constants) *by the positions and nature of their singularities*. This leads to the question of the construction of functions with prescribed properties on a given Riemann surface. Here the topology of the surface is of decisive importance. The surface T is decomposed by means of n crosscuts into a system of m simply connected surface pieces. The number $n - m$, which is independent of the manner of decomposition, is called the order of connectivity of T [**1**, 10–11]; incidentally, in modern terms, this number is equal to the negative of the Euler characteristic of T.

In order to construct appropriate functions on T, Riemann uses a variational principle. (He called it later the Dirichlet principle because he came to know similar procedures in Dirichlet's lectures, and the historically unjustified name stuck.) First T is made into a simply connected surface T^* by means of crosscuts. Then, subject to suitable boundary conditions, the integral

$$\int\left[(u_x - v_y)^2 + (u_y + v_x)^2\right] dx\,dy$$

†*Translator's note.* Reprinted from "*Bernhard Riemann 1826–1866: Turning Points in the Conception of Mathematics*," by Detlef Laugwitz, Translated by Abe Shenitzer. This article is an excerpt (Section 1.2.2, pp. 108–110 and Section 1.2.5, pp. 124–130) from the author's book *Bernhard Riemann*, published by Birkhäuser Verlag in 1996. References such as Article 20 or §20 are to sections of Riemann's dissertation.

is minimized on this surface. If there are singularities to be taken into consideration, then the integral is somewhat modified. With the possible exception of the boundary of T^*, the pair of functions u, v associated with the minimum is a holomorphic function $f = u + iv$. It should be noted that the functional values on the two edges of a crosscut need not coincide; jumps ("periods") may occur.

The paper ends with an application of these methods to the Riemann Mapping Theorem. This theorem asserts that in certain cases the topological equivalence of two surfaces or regions implies their conformal equivalence, i.e., the existence of a conformal mapping between them. Here the theorem is first stated for regions in the complex plane that are homeomorphic to a circular disk.

We will examine the individual key words while considering further developments in the work of Riemann and others.

We explain briefly, in modern terms, the form of inference Riemann learned from Dirichlet. Let $I(\varphi, \psi)$ be the integral of $\varphi_x \psi_x + \varphi_y \psi_y$ over a region G and let $J(\varphi) = I(\varphi, \varphi)$. Let η be a function that vanishes on the boundary ∂G of G.

$$J(\varphi + t\eta) = J(\varphi) + 2tI(\varphi, \eta) + t^2 J(\eta)$$

implies that if $J(\varphi) \leq J(\varphi + t\eta)$ is to hold for all t, then we must have $I(\varphi, \eta) = 0$. Put $\Delta\varphi = \varphi_{xx} + \varphi_{yy}$. Our last result, the vanishing of η on ∂G, and the Gauss integral formula (Gauss' theorem) imply that

$$0 = \int_{\partial G} (\varphi_x \eta \, dy - \varphi_y \eta \, dx) = \int_G (\Delta\varphi)\eta \, dF + I(\varphi, \eta) = \int_G (\Delta\varphi)\eta \, dF.$$

Since this holds for every η, it follows that $\Delta\varphi = 0$. In other words, a function that minimizes $J(\varphi)$ is a solution of $\Delta\varphi = 0$. To be sure, the argument does not prove the *existence* of such a function, and this elicited justified criticism.

It is relatively easy to prove the uniqueness of the solution of the boundary-value problem. If ψ were another solution, then $\eta = \varphi - \psi$ would vanish on ∂G. Moreover,

$$J(\varphi) = J(\psi) + 2I(\psi, \eta) + J(\eta)$$

and

$$I(\psi, \eta) = \int_{\partial G} \eta(\psi_x \, dy - \psi_y \, dx) - \int_G (\Delta\psi)\eta \, dF = 0.$$

But then

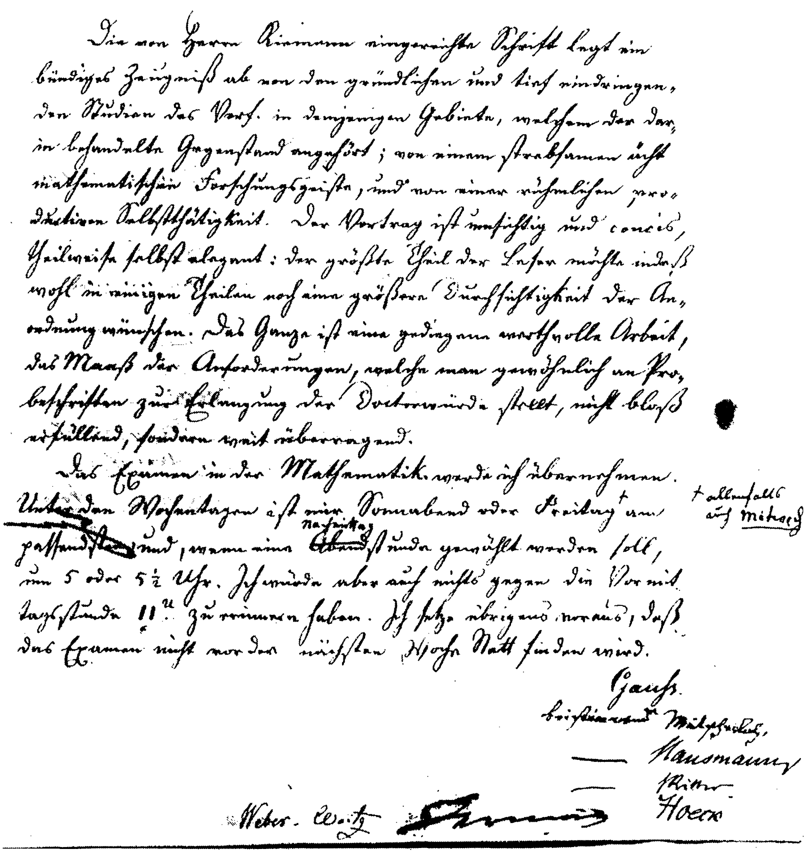
$$J(\varphi) = J(\psi) + J(\eta) \geq J(\psi).$$

In view of the minimality of $J(\varphi)$, the inequality sign in the last expression must be replaced by an equality sign. But then $J(\eta) = 0$, i.e., $\eta_x = \eta_y = 0$. Since $\eta = 0$ on ∂G, it follows that $\eta = 0$, and therefore $\psi = \varphi$ throughout G.

The effect of the dissertation. Today we are inclined to regard Riemann's dissertation as one of the most important achievements of 19th-century mathematics, but its immediate effect was rather slight. We saw that in the second part of Article 20 Riemann himself emphasized just one principle, namely the determination of a function by as few data as possible and the elimination of expressions as definitions of functions. Given its vague formulation, this principle must have struck his contemporaries as neither new nor interesting. Riemann was as restrained in his statement as he was in the specification of his sources.

The first person who had to read the paper carefully was the referee for the Göttingen faculty, that is, Gauss. His report read as follows: "The paper submitted by Herr Riemann is a concise testimony to its author's thorough and penetrating studies of the area to which the subject treated therein belongs; of a diligent and

ambitious, truly mathematical spirit of investigation, and of praiseworthy and fertile independence. The report is prudent and concise, and in places even elegant; nevertheless, most readers might well wish for even greater transparency of arrangement in some of the parts. Taken in its entirety, it is a solid and valuable work which not only meets the requirements usually set for test papers for the attainment of the doctorate but exceeds them by far."


Die von Herrn Riemann eingereichte Schrift legt ein bündiges Zeugniß ab von den gründlichen und tief eindringenden Studien des Verf. in demjenigen Gebiete, welchem der darin behandelte Gegenstand angehört; von einem strebsamen ächt mathematischen Forschungsgeiste, und von einer rühmlichen productiven Selbstthätigkeit. Der Vortrag ist umsichtig und concis, theilweise selbst elegant: der größte Theil der Leser möchte indeß wohl in einigen Theilen noch eine größere Durchsichtigkeit der Anordnung wünschen. Das Ganze ist eine gediegene werthvolle Arbeit, das Maaß der Anforderungen, welche man gewöhnlich an Probeschriften zur Erlangung der Doctorwürde stellt, nicht bloß erfüllend, sondern weit überragend.

Das Examen in der Mathematik werde ich übernehmen. Unter den Wochentagen ist mir Sonnabend oder Freitag + am passendsten, und, wenn eine ~~Abend~~ nachmittag stunde gewählt werden soll, um 5 oder 5½ Uhr. Ich würde aber auch nichts gegen die Vormittagsstunde 11ᵘ zu erinnern haben. Ich setze übrigens voraus, daß das Examen nicht vor der nächsten Woche statt finden wird.

+ allenfalls auch Mittwoch

Gauß.

beistimmend Mitscherlich,
— Hausmann,
— Ritter.
Hoeck
Weber. Waitz

Figure 1. Gauss' testimonial on Riemann's dissertation

If one has a certain amount of experience with evaluations and forgets for a moment that here the *princeps mathematicorum* is writing about a person destined to become probably the most distinguished of his students, then one gets the following impression. The referee recognizes that the author has penetrated deep into a highly specialized field and has done this with great diligence, independently, and without the referee having to suggest the topic to him. There is no mention of the author's new ideas, of the solution of problems, or of new methods, but it is recognized that he may well be showing signs of independent research activity. The presentation is terse, elegant only in spots, and on the whole not clear enough. An objective reader must wonder what was the basis for the "Doktorvater's" (doctoral adviser's) very positive overall evaluation stated in the last sentence. Riemann wrote to his brother:

> When I visited Gauss he had not yet read my paper, but he told me that for years he had been preparing a paper (and is occupied with this right now) whose subject is the same, or partly the same, as the one I am treating

(Incidentally, this passage was quoted by Schering in his memorial address in 1866 [**2**, 835].) So far, no one has been able to find any indication that Gauss had discussed with Riemann the contents of his paper or had given him any hints or suggestions. Riemann would have reported such things. After all, he mentioned the rather disappointing conversation with Gauss which comes down, more or less, to this: right now I happen to be writing on a related topic, but your paper has not interested me enough that I should immediately and eagerly plunge into it.

Some (e.g., Remmert [**6**, Band 2, 158]) think that the old Gauss was "chary of praise" ("lobkarg"). But what argues against this is the fact that a few years earlier he had praised young Eisenstein to the skies. We will make no guesses about the great Gauss' admittedly baffling behavior toward Riemann.

We summarize the essential mathematical concerns that originated in Riemann's dissertation.

(1) The idea of a Riemann surface. Here, for the first time, the domain of definition of a function becomes one of the data that determine it. The complex plane is compactified by the addition of a single point ∞, the Riemann surfaces over it are precisely defined, the connectivity number is introduced and recognized as a topological invariant. (Complex) analysis is carried out not locally but on manifolds, which are compact in the case of algebraic functions. Local representability (by power series) is proved but is of secondary importance.

(2) In addition to poles, branch points are recognized as characteristic types of singularities, and the local series expansions in terms of (negative or fractional) powers are rigorously justified (Article 13/14, [**1**, 24–27]).

(3) The existence (together with the continuity) of $f'(z)$ is equivalent to the Cauchy–Riemann differential equations (together with the continuity of the occurring partial derivatives) and to the conformal character of f. It is also equivalent to the local expandibility, which implies the existence of all derivatives. (Holomorphic or analytic functions.)

(4) The transformation of surface integrals into line integrals is a tool for proving theorems (Articles 7–12, [**1**, 12–24]) of the "Cauchy type."

(5) The ("Dirichlet") principle of the existence of a function that minimizes a surface integral is used to solve boundary-value problems by means of holomorphic functions.

(6) The Riemann Mapping Theorem is a consequence of (5).

The response of contemporaries was amazingly slight; hardly any of the more than 500 titles in Purkert's list covering the period from 1851 to 1891 ([**2**, 869–895]) and relevant to Riemann's dissertation appeared before his death. This is all the more surprising if we keep in mind that two of Riemann's papers that presented the ideas of his dissertation in greater detail and applied them to the solution of problems appeared in 1857. Things were no different when it comes to textbooks. For example, Heinrich Weber's *Elliptische Functionen* of 1891 contains nothing relating to Riemann. Thus one can hardly speak of a significant impact of Riemann's ideas during his lifetime and in the first 25 years after his death. In the subsequent sections we will examine the question of the very special directions in which Riemann influenced research and the question of which elements of his essential ideas failed initially to attract attention.

Let us return to the year of the composition of the dissertation. Jacobi died on 18 February 1851. Dirichlet pushed Riemann in another direction, which led to his habilitation paper on trigonometric series. Representatives of the algorithmic

direction could hardly be expected to approve of Riemann's dissertation. Eisenstein died on 11 October 1852 and Weierstrass had not yet appeared on the scene. The French mathematicians, whose contributions were not explicitly acknowledged in the dissertation, could at best be expected to recognize the concept of a Riemann surface as new. At the same time, they viewed it as too complicated and superfluous. Moreover, Cauchy's students soon got used to working with complex functions in the complex plane in much the same way as Cauchy, who had used complex formulations for his integral theorems and for his method of residues as early as 1831. They must have regarded the method of real partial differential equations as a backward step. At the time doubly periodic functions were in fashion, and they could be dealt with without the use of Riemann surfaces.

Of course, in time the six previously listed key issues associated with the dissertation exerted a powerful effect. What follows is a survey describing this effect.

The effect of (6) was later especially notable in applied mathematics. For a disk, the first boundary-value problem for the potential equation $u_{xx} + u_{yy} = 0$ is solved by the Poisson integral, which expresses the function u in terms of its boundary values. Since the differential equation is invariant under conformal mappings, we obtain a solution of this problem for any simply connected region bounded by a curve by mapping the disk conformally onto this region. But this is just an existence statement, and Riemann's theorem does not directly yield a formula representing the solution. Such representations were eventually obtained for regions of practical importance by H. A. Schwarz, E. B. Christoffel, and others.

The mapping theorem became effective in many respects independently of applications and of the other objectives and contents of the dissertation. It is an instance of Riemann's novel view of mathematics. For one thing, it illustrates the fruitfulness of the notion that functions are simply mappings. For another, it is a global proposition; all Gauss could prove was the conformal equivalence of small pieces of surfaces. Finally it was one of the deeper existence theorems to emerge after Cauchy's existence theorems about solutions of differential equations. For adherents of algorithms this was an unusual type of proposition; indeed, *they* took note of transformations only if they were associated with effective formulas. It is also noteworthy that the theorem shows that the theory of functions on a simply connected region with boundary is completely independent of the special choice of region. When investigating a special class of functions we can choose a convenient special region, say the upper halfplane.

Riemann's sketch of a proof in §21 is cryptic, and not just because of his use of the Dirichlet principle. Efforts to fully justify the idea of his proof failed. Given the importance of the theorem for applications, this failure stimulated attempts to develop new methods of proof. These remarks also apply to the uniformization theorem, which generalizes Riemann's mapping theorem. The geometric formulation promoted the acceptance of the notion of a Riemann surface. Riemann himself spoke [**1**, 40] of "geometric clothing" ("geometrische Einkleidung") used for "illustration and more convenient wording" (zur "Veranschaulichung und bequemeren Fassung"), formulations hardly ever encountered elsewhere in his writings. The use of complex methods for the computation of definite integrals opened up a new field for the applicability of complex function theory, and that is why complex analysis became a fixed component of the mathematical education of physicists and engineers. As for mathematics itself, the question of admissible boundaries of simply connected regions provided essential impulses for the evolution of point set theory.

For the effects of the dissertation in the first fifty years after Riemann, see [**5**]. For later developments see [**6**, Band 2, 157–163]. We recommend [**3**] and especially [**4**], a book saturated with Riemann's style of thinking. It is safe to say that, even had Riemann's dissertation consisted of just the mapping theorem, its influence would ultimately have been considerable.

The effect of (5) was unexpected. Riemann's justification of the existence of a minimal solution is inadequate. This was noted by Weierstrass, whose 1870 criticism was devastating and seemed to destroy the very basis of Riemann's justification of complex analysis. But this had also very positive consequences.

One consequence was that people tried, successfully, to prove the relevant results without using the Dirichlet principle. Actually they would have tried to find such proofs regardless of doubts about this principle. Such attempts reflect the wish to construct complex function theory in a "purely complex" way and to avoid the use of tools from real analysis, functions u and v of two real variables x and y. This too was achieved. Incidentally, this does not signify the rejection of Riemann's development of function theory. In view of its conceptual basis, it is closer to our way of thinking than is, say, the Weierstrass approach.

Another consequence of the criticism directed at Riemann's justification of the Dirichlet principle was even more important than the first one. Since there were no counterexamples and the principle itself was believable, people felt that it must be provable. Hilbert obtained a proof after 1900, and in doing so developed the so-called direct methods of the calculus of variations, which avoid the detour through the partial differential equations associated with the variational problem. One begins instead with a sequence of functions for which the values of the integral, or more generally of the functional, to be minimized approximate the infimum. One must show that the space of admissible functions has a compactness property which justifies the conclusion that a subsequence converges to a function for which the functional takes on its minimum. In this way a method was developed that not only saved the Dirichlet principle but has progressively become more important in the 20th century.

But let us go back briefly to the attempts to avoid the Dirichlet principle. Much was achieved by H. A. Schwarz and C. Neumann. As for the mapping theorem, the conclusive result was obtained independently by Poincaré and by Koebe in 1907. It asserts that every simply connected Riemann surface is holomorphically equivalent to one of following three surfaces: $\mathbb{C} \cup \{\infty\}$ (the number sphere or complex projective straight line), $\mathbb{C}$ (the number plane or complex straight line), or the open disk $|z| < 1$. The key that leads one to this group of problems in the literature is the uniformization theorem. This problem and its easy-to-formulate answer were almost obvious to Riemann, but half a century was needed to obtain it.

We do not know whether Riemann expected a stronger response. After all, he did say

> However, we now refrain from the realization of this theory...for we rule out, at present, consideration of an expression of a function

He set aside for a few years the task of investigating concrete functions and classes of functions, and tackled it in connection with lectures devoted to these matters. Of course, this did not happen during his first year as university instructor.

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Snow and Ice and Numbers

It seems necessary to explain my claustrophobia to him.

"Do you know what the foundation of mathematics is?" I ask. "The foundation of mathematics is numbers. If anyone asked me what makes me truly happy, I would say: numbers. Snow and ice and numbers. And do you know why?"

He splits the claws with a nutcracker and pulls out the meat with curved tweezers.

"Because the number system is like human life. First you have the natural numbers. The ones that are whole and positive. The numbers of a small child. But human consciousness expands. The child discovers a sense of longing, and do you know what the mathematical expression is for longing?"

He adds cream and several drops of orange juice to the soup.

"The negative numbers. The formalization of the feeling that you are missing something. And human consciousness expands and grows even more, and the child discovers the in between spaces. Between stones, between pieces of moss on the stones, between people. And between numbers. And do you know what that leads to? It leads to fractions. Whole numbers plus fractions produce rational numbers. And human consciousness doesn't stop there. It wants to go beyond reason. It adds an operation as absurd as the extraction of roots. And produces irrational numbers."

He warms French bread in the oven and fills the pepper mill.

"It's a form of madness. Because the irrational numbers are infinite. They can't be written down. They force human consciousness out beyond the limits. And by adding irrational numbers to rational numbers, you get real numbers."

I've stepped into the middle of the room to have more space. It's rare that you have a chance to explain yourself to a fellow human being. Usually you have to fight for the floor. And this is important to me.

"It doesn't stop. It never stops. Because now, on the spot, we expand the real numbers with imaginary square roots of negative numbers. There are numbers we can't picture, numbers that normal human consciousness cannot comprehend. And when we add the imaginary numbers to the real numbers, we have the complex number system. The first number system in which it's possible to explain satisfactorily the crystal formation of ice. It's like a vast, open landscape. The horizons. You head toward them and they keep receding. That is Greenland, and that's what I can't be without! That's why I don't want to be locked up."

Smilla's Sense of Snow, by Peter Høeg, translated by Tiina Nunnally
Dell Publishing, New York, 1994, pp. 121–122

Contributed by Evan J. Romer, Windsor, NY

PROBLEMS AND SOLUTIONS

Edited by **Gerald A. Edgar, Daniel H. Ullman,** and **Douglas B. West**

with the collaboration of Paul T. Bateman, Mario Benedicty, Paul Bracken, Duane M. Broline, Ezra A. Brown, Richard T. Bumby, Glenn G. Chappell, Randall Dougherty, Roger B. Eggleton, Ira M. Gessel, Bart Goddard, Jerrold R. Griggs, Douglas A. Hensley, Richard Holzsager, John R. Isbell, Robert Israel, Kiran S. Kedlaya, Murray S. Klamkin, Fred Kochman, Frederick W. Luttmann, Vania Mascioni, Frank B. Miles, Richard Pfiefer, Cecil C. Rousseau, Leonard Smiley, John Henry Steelman, Kenneth Stolarsky, Richard Stong, Charles Vanden Eynden, and William E. Watkins.

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the inside front cover. Submitted problems should include solutions and relevant references. Submitted solutions should arrive at that address before October 31, 1999; Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An acknowledgement will be sent only if a mailing label is provided. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

10732. *Proposed by M. N. Deshpande, Nagpur, India.* Let n and k be positive integers with $k < n$. Select a permutation π of n objects at random, and let the random variable X_k denote the number of objects that lie in cycles of π of length less than or equal to k. Find the expected value and the variance of X_k.

10733. *Proposed by Sung Soo Kim, Hanyang University, Ansan, Korea.* Let $\{E_\alpha\}_{\alpha\in\Omega}$ be a partition of the unit interval $I = [0, 1]$ into nonempty sets that are closed in the usual topology. Is it possible that
(**a**) Ω is uncountable and E_α is uncountable for each $\alpha \in \Omega$?
(**b**) Ω is uncountable but E_α is countably infinite for each $\alpha \in \Omega$?
(**c**) Ω is countably infinite?

10734. *Proposed by Floor van Lamoen, Goes, The Netherlands.* Let ABC be a triangle with orthocenter H, incenter I, and circumcenter O. Let $[P, r]$ denote the circle with center P and radius r. Show that the radical center of $[A, CA + AB]$, $[B, AB + BC]$, and $[C, BC + CA]$ is the point obtained by reflecting H through O and then reflecting the result through I.

10735. *Proposed by Gustavus J. Simmons, Sandia Park, NM.* If L_n is the n-by-n matrix with i, j-entry equal to $\binom{i-1}{j-1}$, then $L_n^2 \equiv I_n \bmod 2$, where I_n is the n-by-n identity matrix. Show that if R_n is the n-by-n matrix with i, j-entry equal to $\binom{i-1}{n-j}$, then $R_n^3 \equiv I_n \bmod 2$.

10736. *Proposed by Mizan R. Khan, Eastern Connecticut State University, Willimantic, CT.* For a given $n \geq 2$, let $M(n) = \max\{\,|a - b| : a, b \in \{1, 2, \ldots, n\} \text{ and } ab \equiv 1 \bmod n\,\}$.
(**a**) Find a closed-form expression $U(n)$ such that $M(n) \leq U(n)$ for all n, with equality in infinitely many cases.
(**b**) Show that $\lim_{n\to\infty} M(n)/n = 1$.
(**c**)* Prove or disprove that $\lim_{n\to\infty} \log(n - M(n))/\log n = 1/2$.

10737. *Proposed by Hassan Ali Shah Ali, Tehran, Iran.* Let m and n be positive integers with $n \geq 2m$, and let $a_1 \leq a_2 \leq \cdots \leq a_n$ be positive integers such that

$$a_n < m + \frac{1}{2m}\left(\sum_{i=1}^{m}\binom{n}{2i}\binom{2i}{i}\right).$$

Show that there exist two different n-tuples $(\epsilon_1, \ldots, \epsilon_n)$ and $(\delta_1, \ldots, \delta_n)$, with entries 0, 1, and 2, such that $\sum_{j=1}^{n}\epsilon_j = \sum_{j=1}^{n}\delta_j \leq 2m$ and $\sum_{j=1}^{n}\epsilon_j a_j = \sum_{j=1}^{n}\delta_j a_j$.

10738. *Proposed by Radu Theodorescu, Université Laval, Sainte-Foy, PQ, Canada.* For $t > 0$, let $m_n(t) = \sum_{k=0}^{\infty} k^n e^{-t} t^k/k!$ be the nth moment of a Poisson distribution with parameter t. Let $c_n(t) = m_n(t)/n!$. A sequence $a_0, a_1, \ldots$ is *log-convex* if $a_{n+1}^2 \leq a_n a_{n+2}$ for all $n > 0$ and is *log-concave* if $a_{n+1}^2 \geq a_n a_{n+2}$ for all $n > 0$.
(**a**) Show that $m_0(t), m_1(t), \ldots$ is log-convex.
(**b**) Show that $c_0(t), c_1(t), \ldots$ is not log-concave when $t < 1$.
(**c**) Show that $c_0(t), c_1(t), \ldots$ is log-concave when t is sufficiently large.
(**d**)* Is $c_0(t), c_1(t), \ldots$ log-concave when $t \geq 1$?

SOLUTIONS

Moments of Roots of Chebyshev Polynomials

10448 [1995, 360]. *Proposed by Fu-Chuen Chang, National Sun Yat-sen University, Kaohsiung, Taiwan.* Fix a positive integer n. Let $x_i = \cos\big((2i-1)\pi/(2n)\big)$ for $1 \leq i \leq n$, and let $c_k = \frac{1}{n}\sum_{i=1}^{n} x_i^k$ for $k \in \mathbb{N}$. Show that

$$c_k = \begin{cases} 0 & \text{if } k = 1, 3, \ldots, 2n-1; \\ \binom{k}{k/2}2^{-k} & \text{if } k = 0, 2, \ldots, 2n-2. \end{cases}$$

Solution I by Paul Deiermann, Louisiana State University, Shreveport, LA. When $k = 0$ and n is odd, the term for $j = (n+1)/2$ appears as 0^0, which must be taken to be 1 to arrive at the stated formula and our generalization. We show, for arbitrary integers $k \geq 0$, that

$$c_k = \begin{cases} 0 & \text{for } k \text{ odd}, \\ 2^{-k}\sum_{p=-m}^{m}(-1)^p\binom{k}{pn+\frac{k}{2}} & \text{for } k \text{ even}, \end{cases}$$

where $m = \lfloor k/(2n) \rfloor$. The stated problem covers those k for which $m = 0$.

First note that $x_{n+1-j} = -x_j$, so the terms of the sum cancel in pairs when k is odd. We may thus restrict to the case of k even. Since $x_j = \left(e^{i\pi(2j-1)/(2n)} + e^{-i\pi(2j-1)/(2n)}\right)/2$, the binomial theorem and a summation of a finite geometric progression imply

$$\sum_{j=1}^{n} x_j^k = \sum_{j=1}^{n} 2^{-k}\left(e^{i\pi\frac{2j-1}{2n}} + e^{-i\pi\frac{2j-1}{2n}}\right)^k = 2^{-k}\sum_{j=1}^{n}\sum_{q=0}^{k}\binom{k}{q}e^{i\frac{\pi}{2n}(k-2q)}e^{i\frac{2\pi}{n}(q-k/2)j}$$

$$= 2^{-k}\sum_{q=0}^{k}\binom{k}{q}e^{i\frac{\pi}{2n}(k-2q)}\sum_{j=1}^{n}e^{i\frac{2\pi}{n}(q-k/2)j} = 2^{-k}\sum_{q=0}^{k}\binom{k}{q}e^{i\frac{\pi}{2n}(2q-k)}\sum_{u=0}^{n-1}e^{i\frac{2\pi}{n}(q-k/2)u}$$

$$= 2^{-k}\sum_{q=0}^{k}\binom{k}{q}e^{i\frac{\pi}{2n}(2q-k)}\begin{cases} n & \text{if } q - k/2 = pn,\ p \in \mathbb{Z}, \\ \frac{1-e^{i\pi(2q-k)}}{1-e^{i\frac{2\pi}{n}(q-k/2)}} = 0 & \text{if } n \nmid q - k/2. \end{cases}$$

Since k is even, $q - k/2 = pn$ implies $q = pn + k/2$. Then, $0 \leq q \leq k$ gives $-m \leq p \leq m$. Also, in this case, $e^{i\frac{\pi}{2n}(2q-k)} = e^{i\pi p} = (-1)^p$. Thus, we get

$$\sum_{j=1}^{n} x_j^k = 2^{-k}n\sum_{p=-m}^{m}(-1)^p\binom{k}{pn+\frac{k}{2}}.$$

Solution II by Walter Van Assche, Katholieke Universiteit Leuven, Heverlee, Belgium. The x_i are the zeros of the Chebyshev polynomial of the first kind T_n of degree n. The Gauss-Chebyshev quadrature formula has the property that the quadrature weights are constant; thus Gaussian quadrature gives

$$\frac{1}{n}\sum_{k=1}^{n} f(x_i) = \frac{1}{\pi}\int_{-1}^{1} f(x)\frac{dx}{\sqrt{1-x^2}}$$

for every polynomial f of degree at most $2n-1$ (T. J. Rivlin, *Chebyshev Polynomials*, Wiley, 1990, pp. 43–46). Taking $f(x) = x^k$ for $0 \le k \le 2n-1$ then gives

$$c_k = \frac{1}{\pi}\int_{-1}^{1} x^k \frac{dx}{\sqrt{1-x^2}}.$$

By symmetry this integral vanishes when k is odd. When k is even, the symmetry and the substitution $x^2 = t$ gives

$$\int_{-1}^{1} x^k \frac{dx}{\sqrt{1-x^2}} = \int_0^1 t^{\frac{k-1}{2}} \frac{dt}{\sqrt{1-t}}.$$

The latter is Euler's Beta function $B((k+1)/2, 1/2) = \Gamma((k+1)/2)\Gamma(1/2)/\Gamma(k/2+1)$. Now use Legendre's duplication formula $\Gamma(2z) = (2\pi)^{-1/2}2^{2z-1/2}\Gamma(z)\Gamma(z+1/2)$ with $2z = k+1$ and $\Gamma(1/2) = \sqrt{\pi}$ to find the desired results.

Solution III by Franz Peherstorfer, Johannes Kepler Universität, Linz, Austria. For $x \in [-1, 1]$, let $T_n(x) = \cos(n \arccos x)$ and $U_n(x) = \sin((n+1)\arccos x)/\sin(\arccos x)$ denote the degree n Chebyshev polynomials of the first and second kind, respectively. Since $T_n(x) = 2^{n-1}\prod_{i=1}^{n}(x - x_i)$ and $T_n'(x) = nU_{n-1}(x)$, we have

$$\frac{U_{n-1}(x)}{T_n(x)} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{x-x_i} = \sum_{k=0}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n} x_i^n\right)\frac{1}{x^{k+1}} \tag{1}$$

for $|x| > 1$, where the second equality follows from a series expansion of $(1 - x_i/x)^{-1}$.

On the other hand, we have $T_n^2(x) - (x^2-1)U_{n-1}^2(x) = 1$ for all $x \in \mathbb{R}$. Dividing both sides of this equation by $(x^2-1)T_n^2(x)$ gives

$$\left(\frac{1}{\sqrt{x^2-1}} - \frac{U_{n-1}(x)}{T_n(x)}\right)\left(\frac{1}{\sqrt{x^2-1}} + \frac{U_{n-1}(x)}{T_n(x)}\right) = \frac{1}{(x^2-1)T_n^2(x)} = O\left(\frac{1}{x^{2n+2}}\right)$$

as $x \to \infty$. Since $\lim_{x\to\infty} xU_{n-1}(x)/T_n(x) = 1$, this implies

$$\frac{U_{n-1}(x)}{T_n(x)} = \frac{1}{\sqrt{x^2-1}} + O\left(\frac{1}{x^{2n+1}}\right) \tag{2}$$

as $x \to \infty$. Taking the series expansion of $\sqrt{1-x^{-2}}$ in (2) and comparing to the series in (1) gives the desired result.

Editorial comment. Wolfdieter Lang noted that the generating function $\sum_{k\ge 0} c_k z^k$ has been computed explicitly as an elementary function. See W. Lang, On sums of powers of zeros of polynomials, *J. Comp. Appl. Math.* 88 (1998) 237–256 for details and further references.

Solved also by U. Abel (Germany), J. Anglesio (France), G. Bach (Germany), K. L. Bernstein, N. Bhatnagar, J. C. Binz (Switzerland), P. Bracken & S. Dorf (Canada), R. J. Chapman (U. K.), H. Chen, E. Cohen (France), D. A. Darling, K. Diethelm (Germany), C. J. Efthimiou, R. Ehrenborg (Canada), S. M. Gagola Jr., M. E. H. Ismail, N. Komanda, R. L. Lamphere, W. Lang (Germany), J. H. Lindsey II, O. P. Lossers (The Netherlands), A. Pedersen (Denmark), N. Rosenberg, K. Foltz, H.-J. Seiffert (Germany), S. J. Smith (Australia), A. Stenger, R. Stong, M. Vowe (Switzerland), H. Widmer (Switzerland), Anchorage Math Solutions Group, NSA Problems Group, and the proposer.

Indecomposable Numbers

10589 [1997, 362]. *Proposed by Tim Keller, Fair Oaks, CA.* Fix $n \geq 3$, and let S be the set of positive integers congruent to 1 modulo n. A number $m \in S$ is called *indecomposable* if it is not the product of two smaller numbers in S. Problem 2 from the 1977 International Mathematical Olympiad asks for a number that can be expressed as the product of indecomposable numbers in more than one way. Show that the least such number is the product of two numbers each of the form $k(k+n)$.

Solution by the GCHQ Problems Group, Cheltenham, U. K. Define a *clone* to be a number expressible as a product of indecomposable factors in two different ways. Let m be the smallest clone. By the minimality of m, no indecomposable factor can appear in both expressions. Let $an+1$ be the smallest indecomposable factor in either expression, and let $bn+1 = m/(an+1)$. Let $cn+1$ be an indecomposable factor in the other expression, and let $dn+1 = m/(cn+1)$. Thus $m = (an+1)(bn+1) = (cn+1)(dn+1)$.

Since $cn+1$ is indecomposable, $an+1$ does not divide it. Also $an+1$ does not divide $dn+1$, since otherwise $dn+1$ is a smaller clone than m. Therefore $an+1$ is not prime and factors as pq, where $p|(cn+1)$ and $q|(dn+1)$. Both p and q are coprime to n.

Now $p|(an+1)$ and $p|(cn+1)$, so $p|(c-a)n$. Since p is coprime to n, we have $p|(c-a)$, so $c = rp + a$, where $r \geq 1$ since $c > a$. Hence $cn+1 = rpn + an + 1 = rpn + pq = p(rn+q)$. Similarly, $q|(d-a)n$ leads to $dn+1 = q(sn+p)$, where $s \geq 1$. Thus $m = p(rn+q)q(sn+p)$.

Finally, we show that $r = s = 1$. Let $t = p(n+q)q(n+p)$. If $r > 1$ or $s > 1$, then $t < m$, so t must not be a clone. Since $t = pq \times (n+p)(n+q)$ and pq is indecomposable, pq must divide one of the two factors in the factorization $t = p(n+q) \times q(n+p)$. But if $pq|p(n+q)$, then $pq|pn$, and $q|n$, a contradiction since q is coprime to n. An identical argument shows that pq cannot divide $q(n+p)$.

With $r = s = 1$, we have $m = p(n+p) \times q(n+q)$, as desired.

Editorial comment. The proposer and the NCCU Problems Group both noted that pq is not necessarily the smallest composite congruent to 1 modulo n, giving the example $n = 336$, where $336k+1$ is prime for $1 \leq k \leq 3$, $336 \cdot 4 + 1 = 5 \cdot 269$, and $336 \cdot 5 + 1 = 41 \cdot 41$, but $5 \cdot 269(5+336)(269+336) > 41 \cdot 41(41+336)(41+336)$.

Solved also by X. Wang, NCCU Problems Group, and the proposer.

Negatively Correlated Vectors of Signs

10593 [1997, 456]. *Proposed by Donald E. Knuth, Stanford University, Stanford, CA.* A certain matrix has m rows and $n = 1 + k^2$ columns. All entries of the matrix are ± 1, and the dot product of any two columns is less than or equal to 0. Prove that the total number of positive entries in the matrix is at most $\frac{1}{2}m(n+k)$, and construct a matrix that achieves this upper bound.

Solution by GCHQ Problem Solving Group, Cheltenham, U. K. Consider the sum S of the dot products of all pairs of columns. Since each dot product is nonpositive, so is S. If row i has r_i positive entries, then its contribution to the sum is $\binom{r_i}{2} + \binom{n-r_i}{2} - r_i(n-r_i)$, which equals $\left((2r_i - n)^2 - n\right) / 2$.

Substituting $r_i = (n + k + b_i)/2$ leads to

$$S = \frac{1}{2}\sum_{i=1}^{m}\left((k+b_i)^2 - n\right) = \frac{1}{2}\sum_{i=1}^{m}\left((k+b_i)^2 - \left(1+k^2\right)\right) = \frac{1}{2}\sum_{i=1}^{m}\left(2kb_i + b_i^2 - 1\right).$$

Since $S \leq 0$, we obtain

$$\sum_{i=1}^{m} b_i \leq \frac{1}{2k}\sum_{i=1}^{m}(1 - b_i^2).$$

Since $r_i = (1+k+k^2+b_i)/2$ and r_i is an integer, b_i must be odd, and so $1-b_i^2 \le 0$ for all i. Therefore $\sum_{i=1}^m b_i \le 0$. The total number of positive entries in the matrix thus satisfies

$$\sum_{i=1}^m r_i = \sum_{i=1}^m \frac{1}{2}(n+k+b_i) = \frac{m}{2}(n+k) + \frac{1}{2}\sum_{i=1}^m b_i \le \frac{m}{2}(n+k).$$

Achieving the bound requires $\sum_{i=1}^m b_i = 0$, which occurs only when half the rows have $b_i = +1$ and the other half have $b_i = -1$. Thus it is necessary that m be even. One matrix that achieves the bound when $m = 2(n!)$ is formed by taking all $n!$ permutations of a row with $\frac{1}{2}(n+k+1)$ positive entries and all $n!$ permutations of a row with $\frac{1}{2}(n+k-1)$ positive entries. By symmetry, all of the dot products are equal, and their sum is zero; hence each dot product must be zero.

Editorial comment. John H. Lindsey observed that equality in the bound requires m to be divisible by 4. The proposer asked for the smallest number of rows allowing equality to be achieved for a given n. He and Richard Stong independently provided a construction with $m = 2\binom{n-1}{(k+1)k/2}$.

Solved also by R. J. Chapman (U. K.), J. H. Lindsey II, K. McInturff, R. Stong, and the proposer.

n-Tuples Whose Elements Divide Their Sum

10597 [1997, 457]. *Proposed by David Cox, Amherst College, Amherst, MA.* Fix an integer $n \ge 2$, and let $d_1, d_2, \ldots, d_n$ be positive integers with no common divisor greater than 1. Suppose that d_i divides $d_1 + \cdots + d_n$ for $1 \le i \le n$.
(**a**) Prove that $d_1 d_2 \cdots d_n$ divides $(d_1 + \cdots + d_n)^{n-2}$.
(**b**) For each $n \ge 3$, give an example to show that the exponent in part (**a**) cannot be made smaller.

Solution by GCHQ Problems Group, Cheltenham, U. K.
(**a**) Let p be a prime factor of the product $d_1 d_2 \cdots d_n$, and let p^k be the highest power of p dividing any one of the d_i. We have $p^k \mid \sum d_i$, and thus $p^{k(n-2)} \mid \left(\sum d_i\right)^{n-2}$. Since $d_1, \ldots, d_n$ have no common factor greater than 1, some element d_j is not divisible by p. Furthermore, since $p \mid \sum d_i$, at least two summands are not divisible by p. Hence the highest power of p dividing $\prod d_i$ does not exceed $p^{k(n-2)}$. Repeating this for each prime factor shows that $\prod d_i$ divides $(\sum d_i)^{n-2}$.
(**b**) Let $d_1 = 1$, $d_2 = n-1$, and $d_i = n$ for $3 \le i \le n$. Here $\sum d_i = n(n-1)$, which is divisible by each d_i. Since $d_1 = 1$, the greatest common divisor is 1. We have $\prod d_i = n^{n-2}(n-1)$. Since n and $n-1$ are coprime, the smallest power of $n(n-1)$ divisible by $n^{n-2}(n-1)$ is $(n(n-1))^{n-2}$, and thus the exponent cannot be reduced.

Editorial comment. Other examples submitted for part (**b**) by various solvers included

$$d_1 = 1, d_2 = 2, \text{ and } d_i = 3 \cdot 2^{i-3} \text{ for } 3 \le i \le n$$

and

$$d_1 = 1, d_i = 2 \text{ for } 2 \le i \le n-1, \text{ and } d_n = 2n-3.$$

Using Euclid's sequence 2, 3, 7, 43, 1807, ..., the San Jose State Problem Solving Ring gave an example in which $d_1 d_2 \cdots d_n = (d_1 + \cdots + d_n)^{n-2}$. Another use of Euclid's sequence appears in this MONTHLY in the solution of Problem 10532 [1996, 510; 1998, 775], where references are given.

M. J. Knight and the San Jose State Problem Solving Ring each showed that for given n the set D_n of n-tuples $(d_1, d_2, \ldots, d_n)$ satisfying the conditions of the problem is finite. For example, D_2 contains only the pair (1, 1), and D_3 contains only the triples (1, 1, 1), (1, 1, 2), (1, 2, 3), and their permutations. The finiteness of D_n is equivalent to the finiteness

of the set X_n of solutions of $1/x_1 + 1/x_2 + \cdots + 1/x_n = 1$ in positive integers, which was apparently first established by D. R. Curtiss, this MONTHLY 29 (1922) 380–387. A direct bijection between D_n and X_n is obtained by setting $x_j = \left(\sum d_i\right)/d_j$.

Solved also by R. Barbara (Lebanon), D. Beckwith, M. Boase (U.K.), J. Brawner, D. Callan, R. J. Chapman (U. K.), T. Hermann, R. Holzsager, T. Jager, S. A. Jassim (U. K.), M. J. Knight, C. Lanski, J. H. Lindsey II, D. Lorenzini, K. McInturff, R. Padma (India), K. Schilling, R. Stong, A. Tissier (France), SJSU Problem Solving Ring, and the proposer.

Binomial Ratios

10625 [1997, 871]. *Proposed by Olaf Krafft and Martin Schaefer, Technical University Aachen, Aachen, Germany.* For $x > 0$ and $n \in \mathbb{N}$, define

$$a_n = \sum_{i=0}^{2^{n-1}} \binom{2^n}{2i} x^i \Bigg/ \sum_{i=0}^{2^{n-1}-1} \binom{2^n}{2i+1} x^i.$$

Evaluate $\lim_{n\to\infty} a_n$.

Solution I by Nora Thornber, Raritan Valley Community College, Somerville, NJ. Applying the binomial theorem four times, we have

$$a_n = \sqrt{x} \cdot \frac{\left(1+\sqrt{x}\right)^{2^n} + \left(1-\sqrt{x}\right)^{2^n}}{\left(1+\sqrt{x}\right)^{2^n} - \left(1-\sqrt{x}\right)^{2^n}} = \sqrt{x} \cdot \frac{1 + \left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)^{2^n}}{1 - \left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)^{2^n}}.$$

But $\left|(1-\sqrt{x})/(1+\sqrt{x})\right| < 1$, so we conclude that $\lim_{n\to\infty} a_n = \sqrt{x}$.

Solution II by The National Security Agency Problems Group, Fort Meade, MD. Let $p = \sqrt{x}/(\sqrt{x}+1)$ and $q = 1/(\sqrt{x}+1)$, so that $0 < p, q < 1$, $p + q = 1$, and $\sqrt{x} = p/q$. Now consider an experiment consisting of 2^n independent tosses of a coin that is biased to come up heads with probability p. Let E_n (respectively, O_n) be the probability that an even (respectively, odd) number of heads comes up. Set $u_n = u_n(p) = E_n/O_n$. Then

$$\begin{aligned} u_n &= \frac{\sum_{i=0}^{2^{n-1}} \binom{2^n}{2i} p^{2i} q^{2^n-2i}}{\sum_{i=0}^{2^{n-1}-1} \binom{2^n}{2i+1} p^{2i+1} q^{2^n-(2i+1)}} \\ &= \frac{q^{2^n} \sum_{i=0}^{2^{n-1}} \binom{2^n}{2i} (p/q)^{2i}}{q^{2^n} \sum_{i=0}^{2^{n-1}-1} \binom{2^n}{2i+1} (p/q)^{2i+1}} = \frac{\sum_{i=0}^{2^{n-1}} \binom{2^n}{2i} x^i}{\sqrt{x} \sum_{i=0}^{2^{n-1}-1} \binom{2^n}{2i+1} x^{2i+1}}. \end{aligned}$$

Hence $a_n = \sqrt{x} u_n$.

The independence of the various tosses implies $E_{n+1} = E_n E_n + O_n O_n$ and $O_{n+1} = 2E_n O_n$. Therefore

$$u_{n+1} = \frac{E_n^2 + O_n^2}{2E_n O_n} = \frac{1}{2}\left(u_n + \frac{1}{u_n}\right).$$

By the arithmetic-geometric mean inequality, $u_n \geq 1$; hence $u_n \geq (1/2)(u_n + 1/u_n) = u_{n+1}$. Therefore the sequence u_n is decreasing and bounded below; it follows that $L = \lim_{n\to\infty} u_n$ exists, and satisfies $L = (1/2)(L + 1/L)$. Therefore $L = 1$, so we conclude that $\lim_{n\to\infty} a_n = \sqrt{x}$.

Solution III by Ulrich Abel, Fachhochschule Giessen-Friedberg, Friedberg, Germany. We prove the following generalization: For integers $k \geq 1$, $r, s \geq 0$, and real $x > 0$, we have

$$b_n = \sum_{i\geq 0} \binom{kn}{ki+r} x^i \Bigg/ \sum_{i \geq 0} \binom{kn}{ki+s} x^i \longrightarrow x^{(s-r)/k}.$$

In the special case $k = 2$, $r = 0$, $s = 1$, we have $b_{2^{n-1}} = a_n$, and conclude that $a_n \to \sqrt{x}$.

Let z be a primitive kth root of unity. Then the finite geometric sum $\sum_{j=0}^{k-1} z^{ij}$ is k if i is a multiple of k and 0 otherwise. Choose $y > 0$ with $y^k = x$. We obtain

$$\sum_{i\geq 0}\binom{kn}{ki+r}x^i = \frac{1}{k}\sum_{i\geq 0}\binom{kn}{i+r}y^i\sum_{j=0}^{k-1}z^{ij} = \frac{1}{ky^r}\sum_{j=0}^{k-1}z^{-rj}\sum_{i\geq r}\binom{kn}{i}y^i z^{ij}$$
$$= \frac{1}{ky^r}\sum_{j=0}^{k-1}z^{-rj}(1+yz^j)^{kn} + O(n^{r-1}) = \frac{(1+y)^{kn}}{ky^r}\big(1+o(1)\big)$$

as $n \to \infty$, and this identity also holds with s in place of r. Therefore $b_n \to y^{s-r} = x^{(s-r)/k}$ as $n \to \infty$.

Editorial comment. Jean Anglesio noted that when x is a complex number (but not a negative real) the limit is the principal value of the square root of x. When $x < 0$ the limit does not exist.

Solved also by S. A. Ali, K. F. Andersen (Canada), J. Anglesio (France), D. Beckwith, C. Berg (Sweden), J. C. Binz (Switzerland), P. Bracken (Canada), D. Callan, R. J. Chapman (U. K.), J. E. Dawson (Australia), M. N. Deshpande (India), Z. Franco, C. Georghiou (Greece), T. Hermann, V. Hernandez (Spain), J.-H. Kim, R. A. Kopas, O. Kuba (Syria), N. F. Lindquist, J. H. Lindsey II, N. Lord (U. K.), S. Mahajan, D. A. Morales (Venezuela), M. Omarjee (France), M. M. Patnaik, G. Peng, H. Qin, H. Salle (The Netherlands), V. Schindler (Germany), R. Shahidi (Canada), N. C. Singer, A. Sofo (Australia), A. Stenger, D. B. Tyler, M. Vowe (Switzerland), M. Woltermann, Anchorage Math Solutions Group, GCHQ Problems Group, WMC Problems Group, and the proposer.

A Triangle Inequality

10644 [1998, 175]. *Proposed by Mihàly Bencze, Brazov, Romania.* Given an acute triangle with sides of length a, b, and c, inradius r, and circumradius R, prove that

$$\frac{r}{2R} \leq \frac{abc}{\sqrt{2(a^2+b^2)(b^2+c^2)(c^2+a^2)}}$$

Solution by the GCHQ Problems Group, Cheltenham, England. We have

$$a^2 - (b^2+c^2)(1-\cos A) = b^2+c^2-2bc\cos A - (b^2+c^2) + (b^2+c^2)\cos A$$
$$= (b-c)^2\cos A \geq 0,$$

since A is acute. Hence $a^2 \geq (b^2+c^2)(1-\cos A) = 2(b^2+c^2)\sin^2(A/2)$. It follows that $a^2b^2c^2 \geq 8(a^2+b^2)(b^2+c^2)(c^2+a^2)\sin^2(A/2)\sin^2(B/2)\sin^2(C/2)$, and so

$$\frac{abc}{\sqrt{2(a^2+b^2)(b^2+c^2)(c^2+a^2)}} \geq 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

The standard fact $r = 4R\sin(A/2)\sin(B/2)\sin(C/2)$ now yields the required result.

Editorial comment. Several solvers noted that equality holds when the triangle is equilateral and that the result is valid also when the triangle is not acute.

Solved also by J. Anglesio (France), E. Braune (Austria), Z. Čerin (Croatia), J. Melville (Scotland), C. A. Minh, P. E. Nüesch (Switzerland), G. Peng, C. Popescu (Belgium), C. R. Pranesachar (India), S. M. Soltuz (Romania), M. Vowe (Switzerland), R. L. Young, SAS Maths Club (India), and the proposer.

Limit of a Recurrence

10648 [1998, 271]. *Proposed by N. P. Bhatia, University of Maryland, Baltimore County, MD, and W. O. Egerland, Bel Air, MD.* Let $z_1, z_2, \ldots, z_m$ be $m \geq 2$ points in the complex plane, and let $p_1, p_2, \ldots, p_m$ be positive real numbers such that $p_1 + p_2 + \cdots + p_m = 1$. For ω real and $n > m$, let $z_n = (p_1 z_{n-1} + p_2 z_{n-2} + \cdots + p_m z_{n-m})e^{i\omega}$. Show that the sequence $z_1, z_2, \ldots$ converges, and determine its limit.

Solution by the editors. Let $f_k(s) = \sum_{n=1}^{k} z_n s^n$ and $f(s) = \sum_{n=1}^{\infty} z_n s^n$. Since z_{n+1} is a convex combination of $\{z_1, \ldots, z_m\}$, the sequence is bounded and indeed $|z_n| \leq \max\{|z_1|, |z_2|, \ldots, |z_m|\}$, so the radius of convergence for $f(s)$ is positive. From the recurrence relation we have

$$f(s) = f_m(s) + e^{i\omega} \sum_{k=1}^{m} p_k s^k \big(f(s) - f_{m-k}(s)\big).$$

Thus $f(s)$ is a rational function of s, say $f(s) = A(s)/B(s)$.

Assume first that $e^{i\omega} = 1$. Now $B(s) = 1 - \sum_{k=1}^{m} p_k s^k$ has a zero at $s = 1$ since $\sum_{k=1}^{m} p_k = 1$. It is a simple zero since $B'(1) = -\sum_{k=1}^{m} k p_k$ is not zero. But $B(s)$ has no other zeros on or inside the unit disk, since if $|s| \leq 1$, then $|\sum_{k=1}^{m} p_k s^k| \leq \sum_{k=1}^{m} p_k |s^k| \leq \sum_{k=1}^{m} p_k = 1$, with equality only if $|s| = 1$ and all s^k have the same argument. Thus we have a partial fraction expansion $f(s) = A(1)/(B'(1)(s-1)) + C(s)$ where $C(s)$ is a rational function of s with all poles outside the unit disk. The Maclaurin series for $C(s)$ has radius of convergence greater than 1, so its coefficients go to zero, while the Maclaurin series for $A(1)/(B'(1)(s-1))$ has all coefficients equal to $-A(1)/B'(1)$. It follows that

$$\lim_{n\to\infty} z_n = -\frac{A(1)}{B'(1)} = \frac{\sum_{k=1}^{m} p_k \sum_{n=m-k+1}^{m} z_n}{\sum_{k=1}^{m} k p_k}.$$

If $e^{i\omega} \neq 1$, then $B(s)$ has all its roots outside the closed unit disk, so we see that $z_n \to 0$ without a partial fraction argument.

Solved also by D. M. Bradley, B. Burdick, D. Callan, R. J. Chapman (U. K.), G. Keselman, J. H. Lindsey II, V. Lucic (Canada), W. A. Newcomb, P. Szeptycki, E. I. Verriest, and the proposers.

Random Polynomials with Real Roots

10660 [1998, 366]. *Proposed by Colin L. Mallows, AT&T Laboratories, Florham Park, NJ.* Suppose the coefficients of a polynomial are independent Gaussian random variables, each with mean 0. For each $\epsilon > 0$, can the variances be chosen so that all of the zeroes of the polynomial are real with probability at least $1 - \epsilon$?

Solution by Kenneth Schilling, University of Michigan–Flint, Flint, MI. We prove the following slightly stronger claim by induction on n.

Proposition. *Fix $\varepsilon > 0$ and $n \geq 1$. There exist $\sigma_0, \ldots, \sigma_n > 0$ such that, if (a) $a_0, \ldots, a_n$ are independent Gaussian random variables with mean 0, (b) each a_i has variance σ_i^2, and (c) S is the event $\{a_0 + a_1 x + \cdots + a_n x^n = 0$ has n distinct nonzero real solutions $x\}$, then S has probability at least $1 - \varepsilon$.*

Proof. This is obvious for $n = 1$. To complete the induction, let n and $\varepsilon > 0$ be given. Let $\sigma_0^2, \ldots, \sigma_n^2$ be variances as provided by the induction hypothesis applied to $\varepsilon/2$ and n. Let $f(x)$ denote the random polynomial $a_0 + a_1 x + \cdots + a_n x^n$, and let $g(x) = x f(x)$. Then on the event S, the function g has $n+1$ distinct real zeros, the derivative g' has n real zeros $y_1 < \cdots < y_n$, and the numbers $g(y_i)$ are nonzero and alternate in sign. Hence if $|\delta| < \min\{|g(y_1)|, \ldots, |g(y_n)|\}$, then $h(x) = g(x) + \delta$ has $n+1$ distinct real zeros.

Define the random variable $M = \min\{|g(x)| : x \in \mathbb{R}, g'(x) = 0\}$. Since $M > 0$ on S, there exists $\delta > 0$ such that the probability of the event $S \cap \{M \geq \delta\}$ is at least $1 - \varepsilon/2$. Now let b be a Gaussian random variable, independent of $a_0, \ldots, a_n$, with mean 0 and variance σ^2, where σ^2 is chosen so that $|b| < \delta$ with probability at least $1 - \varepsilon/2$. Then the event $S \cap \{M \geq \delta\} \cap \{0 < |b| < \delta\}$ has probability at least $(1 - \varepsilon/2)^2 > 1 - \varepsilon$, and on this event the equation $h(x) = b + x f(x) = b + a_0 x + a_1 x^2 + \cdots + a_n x^{n+1} = 0$ has $n+1$ distinct nonzero real solutions.

Solved also by J. H. Lindsey II, GCHQ Problems Group (U. K.), and the proposer.

REVIEWS

Edited by **Harold P. Boas**
Mathematics Department, Texas A & M University, College Station, TX 77843-3368

Life's Other Secret. By Ian Stewart. Wiley, New York, 1997, xiii + 285 pp., $24.95 hardcover.
The Magical Maze. By Ian Stewart. Wiley, New York, 1998, xii + 268 pp., $24.95 hardcover.

Reviewed by **Dan Schnabel**

In the city where I reside, smaller bookstores are disappearing due to market pressure from book superstores. The selection of mathematics titles available in these new superstores is astonishing both in its magnitude and in its eccentricity. Books on tricks to improve basic arithmetic skills stand next to highly specialized research texts. Scattered among these are the mathematics popularization books. It is not obvious from the increased shelf space whether mathematics is actually becoming more popular, but these two recent books by Ian Stewart certainly further that goal.

Any good high school mathematics teacher recognizes that to learn mathematics, one must appreciate it; and real-world relevance helps students to appreciate mathematics. Beyond high school, the people who toil or play at mathematics are most likely to be those whose appreciation of the subject is well established. Relevance becomes a lesser concern if it remains a concern at all. Good efforts to popularize mathematics must return to the matter of relevance, and the best works do so in a manner that intrigues even those for whom relevance no longer matters. Ian Stewart's books are among the best; they change the way we look at the world.

Or even the way we look at ourselves.

Standing in front of a mirror waving your left hand, you see an image of yourself waving its right hand. This leads to the question: "Why does a mirror reverse left and right, but not top and bottom?" *The Magical Maze* provides an explanation. For a bilaterally symmetrical object such as the human body, the image created by reflection in a mirror can also be achieved by a 180° rotation in space. Our visual processing system is conditioned to assume that the image results from a rotation, because in the real world it is possible for us to rotate objects manually, but it is not possible for us to reflect them. A left shoe will always be a left shoe, no matter how we manipulate it before our eyes. Seeing symmetry broken by the waving of one hand, we still assume that the image is the result of a rotation—the rotation of a person waving the other hand.

Stewart points out that top and bottom are not switched if you turn the mirror on its side, but he does not consider the more intriguing case in which you lie on your side. Lie on your right side in front of a mirror so that your left leg is on top and your right leg is on the bottom. The image in the mirror has the left leg on the bottom and the right leg on top. Ignoring anything else that might appear in the mirror, we can no longer distinguish whether the mirror has switched left and right or switched top and bottom.

Stewart's ideas and presentation frequently inspired me, even required me, to think beyond his writing, as when I considered the following thought experiment. Imagine an intelligent creature having both left-right symmetry and top-bottom symmetry. (How many eyes would such a creature have?) Would this creature think that mirrors switch left and right, or top and bottom? I suspect that, even though its image can be achieved through two different rotations (as well as through reflection), the creature would still see the mirror as swapping left and right, because of the uniqueness of the vertical orientation it learns from the pull of gravity. What, then, is the role of gravitation in the way our own visual conditioning interprets mirror images? I found myself wondering what Stewart would have to say about this.

Symmetry is a recurring motif in both these books, and the books themselves are somewhat symmetrical. *The Magical Maze* is a mathematics book in which one encounters some biology, while *Life's Other Secret*, subtitled *The New Mathematics of the Living World*, is essentially a biology text in which one encounters some mathematics.

How much mathematics *Life's Other Secret* can be said to contain depends on what one regards as mathematics. Stewart would like readers to acknowledge a broad meaning for mathematics and to recognize a large role for mathematics in the biological sciences.

The title *Life's Other Secret* refers to the idea that genetics and DNA do not provide the complete picture for life on earth. Stewart suggests a different role for genes:

> The cell carries out its genetic instructions; the laws of physics and chemistry produce certain results, and when you put the two together, you get an organism.

Consequently, an understanding of the laws of physics and chemistry and of the underlying mathematics is equal in importance to genetics in the understanding of life.

Given the symmetrical relationship between the two books, it is no surprise that Stewart's best examples of the intersection of biology and mathematics are addressed in both books.

Each of the books contains a discussion of why the number of petals on most flowers is a Fibonacci number. The question is quickly reduced to the arrangement of tissue called primordia at the tip of a plant shoot. As a plant grows, these primordia appear in places that are determined by the need to be closely packed around a circle. The best packings occur when the primordia are separated by an irrational multiple of 360°, because rational multiples of 360° would generate "spokes" of the primordia. The theory of continued fractions can be used to show that the golden ratio $\phi = (1 + \sqrt{5})/2$ is the "most irrational" number. Measurements confirm that primordia are usually positioned around a "generative spiral" separated by an angle that, measured externally, is approximately $360/\phi$ degrees; measured internally this is approximately 137.5°.

The most intriguing aspect of this discussion is that nature is consistent with number theorists on the matter of determining "how irrational" an irrational number is, at least in the case of the golden ratio. It is also interesting that nature appears unwilling to settle for anything other than the most irrational number. Reading Stewart left me wondering what sort of packings would result from irrational numbers that are not the most irrational.

Stewart focuses on the sunflower plant, the large head of which clearly demonstrates the packing problem. He suggests that the position of primordia provides a

nearly complete explanation of the pattern of spirals on the head of a sunflower plant. Missing are the mathematical details explaining why the number of clockwise spirals and the number of counterclockwise spirals that we perceive are two consecutive Fibonacci numbers. We read only that it is because of the close relationship between Fibonacci numbers and the golden ratio, but I would like to have been shown more of the mathematics explaining how the number of spirals we see results from the separation on a "generative spiral". Moreover, Stewart never clarifies the connection between the actual number of petals and the position of primordia—the connection between "how many to arrange" and "how to arrange them".

Both of these books will frequently frustrate the more mathematically minded reader, as they often stop short of the interesting, nitty-gritty mathematical details. This is not necessarily a bad thing: the books are quite effective at whetting one's appetite for more mathematics. But they are not always good at indicating where to turn for more details. *The Magical Maze*, which is the more mathematical of the two books, does provide more details in the endnotes, but they are referred to in the text in a manner that, while intended to be consistent with the maze metaphor of the book, is awkward. The endnotes are numbered, but the references to them are not, so unless you always read all the endnotes, it is unclear which one is being cited.

On the matter of mathematical details, *Life's Other Secret* is the weaker, more frustrating, of the books, as it is essentially a book of biology. It talks about mathematics without actually including a great deal of traditional mathematics. When Stewart simplifies explanations with expressions such as "The mathematical machinery reveals...," I cannot help wanting to see the machinery in action, not just the results. While there are complicated details of how a tobacco virus develops, nowhere is there mathematics of a comparable level of difficulty.

How much mathematics should be included in a popular mathematics book? This is one of the fundamental questions facing writers in the genre. Certainly *The Magical Maze* is a book in which mathematicians will feel at home. It contains a great deal of mathematics, and it attempts explanations that are unusually complex for a book of its type. Its level of sophistication led me to speculate on the possibility that it succeeded in being published primarily because it was written to accompany the televised 1997 Christmas Lectures of the Royal Institution of Great Britain. Not that I mean by this to disparage the book itself; rather, I wonder about the system that makes books of this calibre a rare occurrence.

No doubt readers with only basic high school mathematics training will find *The Magical Maze* difficult going, made more so by an uncharacteristically high number of mistakes in the explanations and diagrams. While some mistakes are typographical and others appear to be printing errors, there are genuine calculation errors as well. Such challenges to comprehensibility may prevent this book from achieving its popularization aims, but do not significantly diminish its quality.

Meritorious as these two books are, publishers and the general public only seem to readily embrace more superficial books, such as Stewart's earlier work *Nature's Numbers*. The dust jacket for *The Magical Maze* includes *Nature* magazine's praise of *Nature's Numbers*:

> Stewart achieves what other popular mathematics writers merely strive for: an accurate, informative portrayal of contemporary mathematics without a single equation in sight.

As much as I liked *Nature's Numbers*, I hope that other popular mathematics writers are not striving to eliminate equations from their books. The ability to

express abstract ideas in the form of equations is a cornerstone of mathematics. The near-taboo status of equations in popular books is irksome. Although doing mathematics and writing about mathematics are two different things, books that avoid equations altogether are not popularizing mathematics as it truly is, but are merely making mathematics marketable.

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An Introductory Course in Commutative Algebra. By A. W. Chatters and C. R. Hajarnavis. Oxford University Press, 1998, viii + 144 pp., $35 softcover, $75 hardcover.
Introduction to Algebra. By Peter J. Cameron. Oxford University Press, 1998, x + 295 pp., $32 softcover, $65 hardcover.

Reviewed by **Cynthia Woodburn**

Driving home last night, I was unhappy with the particular selection playing on my favorite classical radio station, so I switched to my second favorite classical radio station. Imagine my surprise to hear a voice discussing mathematics, and even more surprisingly, discussing the new pop fascination with mathematics. Evidently, there is a trend in pop culture towards the notion that "math is cool". There is even a cologne for men available now named "Pi". The two books under review may not make it onto any popular best-seller lists, be made into movies, or have colognes named after them, but both could be useful in helping students to appreciate that "abstract algebra is cool".

An Introductory Course in Commutative Algebra is a "lean and lively" introduction to commutative algebra with a definite number-theory perspective. Many examples are number theoretic in nature, and number theory is used frequently to motivate new concepts. For example, the chapter on ruler and compass constructions includes a discussion of the connection between Fermat primes and the constructibility of regular n-gons. Written for use by undergraduates, the book is appropriate for a second-semester course in abstract algebra. Its prerequisites include knowledge of equivalence relations, some elementary group theory such as Lagrange's Theorem, and some basic linear algebra. With caution being used at the places where some elementary group theory is assumed (for instance, Chapter 10 on finite cyclic groups and finite fields), the book could be used as a text for a first-semester abstract algebra course; it would be especially good for a class composed of secondary mathematics education majors.

The book begins with an introductory chapter on rings. Since the focus is on commutative algebra, a ring is defined as a commutative ring with identity. For those more familiar with a ring not necessarily being commutative or having a multiplicative identity, some minor adjustments in thinking must be made. For example, if one defines a ring in this fashion, then ideals are not typically subrings, and the even integers do not form a subring of the integers. Chapters 2 and 3 cover Euclidean rings and the highest common factor. I was disappointed to find that the

Euclidean algorithm is not included (although it is mentioned on p. 39 in Chapter 6). The optional Chapter 4 uses "the ring of Gaussian integers to prove one of the most famous theorems in number theory: every positive integer is the sum of four squares." Next are chapters on the traditional topics of fields and polynomials, unique factorization domains, the field of quotients of an integral domain, factorization of polynomials, fields and field extensions, finite cyclic groups and finite fields, and algebraic numbers. Chapter 12 on ruler and compass constructions contains instructions for classical constructions, such as bisecting angles and line segments and dropping perpendiculars, along with the algebra of constructible numbers. The three impossible constructions from antiquity are discussed, as well as the proof by Gauss that the regular 17-gon is constructible (the longest proof of the text). The final three chapters cover homomorphisms, ideals and quotient rings (some familiarity with quotient groups is assumed), principal ideal domains and a method for constructing fields, and finite fields.

The text is extremely readable with a very concrete approach. It is evident that the authors strove to make the text understandable by undergraduate students. Comments are included to explain the significance of results. Definitions are often repeated when terminology is reused, and difficult concepts are explained in everyday language. The book abounds with examples, many of which include actual numbers, something students who are struggling with proofs and abstraction will appreciate. Most of the exercises are straightforward. Some are very easy, such as proving that every field is an integral domain (which appears as Exercise 5.1 and Exercise 9.2). More challenging exercises have hints or contain sketches of a solution with the details to be filled in by students. Answers to selected exercises can be found in the back of the book, although there is no notation within the exercise sets to indicate which problems have solutions provided.

The second text under review, *Introduction to Algebra*, is "lively" but not "lean". While the text by Chatters and Hajarnavis contains fifteen short chapters (the shortest—on constructing the field of quotients of an integral domain—is $2\frac{1}{2}$ pages, and the longest—on ruler and compass constructions—is 15 pages), the text by Cameron, twice the length, is organized into eight large chapters with sections and subsections. This book contains more information than can be covered in a year-long algebra course, which allows for flexibility in its use. After an introductory chapter containing preliminary concepts and motivating material about algebra, the book begins with a study of ring theory guided by the familiar example of the integers. Group theory follows. As pointed out by the author in the preface, these two chapters could form the basis for an introductory one-semester course in abstract algebra. Next are chapters on linear algebra and module theory. Chapter 6 is a change of pace: it contains a formal construction of the natural numbers, integers, rational numbers, real numbers (via Cauchy sequences), and complex numbers; also included are algebraic and transcendental numbers. Chapter 7 contains further topics from group theory, ring theory, and field theory, along with other advanced topics not typically found in a book at this level: namely, universal algebra, lattices, and category theory. The final chapter, entitled "Applications", discusses Galois theory (a classical application), and error-correcting codes (a modern application).

Cameron's writing style is very enjoyable and reader-friendly. He uses entertaining verbs such as "whittle" and "blur" and gives many examples throughout the book. Modern and up-to-date analogies help students relate to concepts: for example, rings with special properties are compared to personal computers with

extra features. Connections between concepts are emphasized. The use of certain terminology and notation is explained, and differences in notation are pointed out. For instance, maps and functions are written on the right in the text, and the reader is urged to "remember that not everybody uses this convention!" Solutions to selected exercises are provided in the back of the book. There is no notation within the exercise sets to denote those problems whose solutions are given, but more difficult exercises are marked with an asterisk. Some of the exercises are even marked with two asterisks.

Even though the text is reader-friendly, a high level of rigor is maintained. Kernels are first defined as equivalence relations, polynomials are defined as infinite sequences, and three different proofs of the existence of transcendental numbers are given.

Both texts include some historical background of terminology and results. Cameron's book also includes some interesting mathematical folklore, which piques the interest of many students. For example, he relates that "legend has it that the [irrationality of the square root of 2] was discovered by Hippasos, a member of the Pythagorean Brotherhood; he was expelled from the Brotherhood (or, in some versions, drowned in a shipwreck) to prevent him from revealing the shameful truth that nature contains irrationality." Each text also includes references or sources for further reading. About half of the references in *An Introductory Course in Commutative Algebra* are from the area of number theory. *Introduction to Algebra* has a more extensive and comprehensive list of sources for further reading, with several introductory paragraphs expounding upon the sources.

Missing from both texts are ideas for cooperative learning activities or writing assignments. Cameron's text does have a very nice web page at `http://www.maths.qmw.ac.uk/~pjc/algebra/`. It contains solutions to all of the exercises from the first three chapters of the text in both LATEX and PostScript formats, further material and problems, links to other sites of interest to algebraists, and corrections.

There are many abstract algebra books on the market. A subject search through a popular online book distributor yielded a list of 148 abstract algebra books and a list of 52 books classified under commutative algebra. The books by Chatters and Hajarnavis and by Cameron are fine additions to the collection of abstract algebra books available for use at the undergraduate level, and each in its own way does a great job of advancing the notion that "abstract algebra is cool".

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TELEGRAPHIC REVIEWS

Edited by **Arnold Ostebee**

with the assistance of the Mathematics Departments of Carleton, Macalester, and St. Olaf Colleges

Telegraphic Reviews are designed to alert readers in a timely manner to new books appropriate to mathematics teaching and research. Special codes classify reviews by subject area and appropriate use:

T : Textbook	*P : Professional Reading*	*1–4: Semester*
C : Computer Software	*L : Undergraduate Library*	*** : Special Emphasis*
S : Supplementary Reading	*13: Grade Level*	*?? : Questionable*

Readers are advised that price information is subject to change. Selected books receive a second, more extensive review in the *Monthly*.

Books submitted for review should be sent to *Book Reviews Editor, American Mathematical Monthly, St. Olaf College, 1520 St. Olaf Avenue, Northfield, MN 55057-1098.*

Mathematics Appreciation, T(13: 1, 2). *Mathematics: A Practical Odyssey, Third Edition.* David B. Johnson, Thomas A. Mowry. Brooks/Cole, 1998, xx + 835 pp, $66.95. [ISBN 0-534-35075-5] Revisions include new section on right angle trigonometry, appendix on dimensional analysis, optional subsections utilizing graphing calculator capabilities. (*Second Edition*, TR, November 1995.) KES

Education, P*, L. *Confronting the Core Curriculum: Considering Change in the Undergraduate Mathematics Major.* Ed: John A. Dossey. MAA Note Series No. 45. MAA, 1998, xii + 136 pp, $38.50 (P). [ISBN 0-88385-155-5] Proceedings of the 1994 "West Point Core Curriculum in Mathematics" conference and a 1995 follow-up workshop. Papers discuss course content issues and student growth goals for the first two years of the undergraduate mathematics curriculum. AO

Discrete Mathematics, T(13–14: 1, 2), L. *Discrete Algorithmic Mathematics, Second Edition.* Stephen B. Maurer, Anthony Ralston. AK Peters, 1998, xix + 894 pp, $59. [ISBN 1-56881-091-1] Republication of 1991 edition with corrections and a few small changes. Greater emphasis on algorithmics, and more sophisticated, than most discrete texts. (*First Edition*, TR, June–July 1991.) KES

Number Theory, P. *Finite Fields: Theory, Applications, and Algorithms.* Eds: Ronald C. Mullin, Gary L. Mullen. Contemp. Math., V. 225. AMS, 1999, x + 243 pp, $49 (P). [ISBN 0-8218-0817-6] Proceedings of a 1997 conference at the University of Waterloo.

Group Theory, P. *The Structure of Compact Groups: A Primer for the Student—A Handbook for the Expert.* Karl H. Hofmann, Sidney A. Morris. Stud. in Math., V. 25. Walter de Gruyter, 1998, xvii + 835 pp, $148.95. [ISBN 3-11-015268-1] A massive, self-contained resource for experts that avoids representation theory and harmonic analysis. JD

Group Theory, P. *Algebraic Groups and their Representations.* Eds: R.W. Carter, J. Saxl. NATO ASI Ser. C, V. 517. Kluwer Academic, 1998, xviii + 374 pp, $173. [ISBN 0-7923-5251-3] 19 articles written by speakers at the 1997 NATO Advanced Study Institute "Modular Representations and Subgroup Structure of Algebraic Groups and Related Finite Groups" held at the Isaac Newton Institute, Cambridge.

Ring Theory, P. *Semidistributive Modules and Rings.* Askar A. Tuganbaev. Math. & Its Applic., V. 449. Kluwer Academic, 1998, x + 352 pp, $157. [ISBN 0-7923-5209-2] Explores the relationship between semidistributive modules and flat, projective, injective, multiplication, and Bezout modules. JD

Ring Theory, T(18), P, L. *Lectures on Modules and Rings.* T.Y. Lam. Grad. Texts in Math., V. 189. Springer-Verlag, 1999, xxiii + 557 pp, $59.95. [ISBN 0-387-98428-3] A follow-up to Lam's *A First Course in Noncommutative Rings* (TR, February 1992); focuses on ring theory in which modules play a central role. Topics: free, projective, injective, and flat modules, rings of quotients, Frobenius rings, Morita theory. Includes 600 exercises. JD

Algebra, T(16–18: 1, 2), L. *Abstract Algebra,*

Second Edition. David S. Dummit, Richard M. Foote. Prentice Hall, 1999, xiv + 898 pp. [ISBN 0-13-569302-0] New material on quadratic integer rings, tensor products of modules and tensor algebras, homological algebra, group cohomology; new chapters on commutative rings and algebraic geometry. (*First Edition*, TR, January 1993.) KES

Algebra, T(18), P, L. *Representations and Cohomology, I: Basic Representation Theory of Finite Groups and Associative Algebras.* D.J. Benson. Stud. in Adv. Math., V. 30. Cambridge Univ Pr, 1995, xi + 246 pp, $29.95 (P); $52.95. [ISBN 0-521-63653-1; 0-521-36134-6] Provides modular representation theoretic background for the study of group cohomology in *Volume II*. *Volume I* concentrates on Auslander–Reiten type representation theory, Burnside rings, and block theory with a cohomological flavor. JD

Algebra, P. *The Monster and Lie Algebras.* Eds: J. Ferrar, K. Harada. Ohio St. Univ. Math. Res. Inst. Public., V. 7. Walter de Gruyter, 1998, x + 252 pp, $248. [ISBN 3-11-016184-2] Proceedings of a 1996 special research quarter at the Ohio State University. In two parts: 9 papers on the Monster; 7 papers on Lie Algebras.

Calculus, S(13), L. *How to Ace Calculus: The Streetwise Guide.* Colin Adams, Joel Hass, Abigail Thompson. WH Freeman, 1998, x + 242 pp, $14.95 (P). [ISBN 0-7167-3160-6] Highly entertaining, useful. Gives advice on such matters as "How to deal with your instructor" (e.g., ask: "Where did you get those ultra cool shoes?") together with a serious, very traditional treatment of calculus topics (e.g., gives much more space to techniques of integration than to numerical techniques; computers are barely mentioned). Certain "light" sections (e.g., "Choosing your instructor") may put some people off. KS

Calculus, T*(13: 3). *Calculus: Single and Multivariable, Second Edition.* Deborah Hughes-Hallett, *et al.* Wiley, 1998, xix + 984 pp, $111.95, [ISBN 0-471-19490-5]; *Calculus: Single Variable, Second Edition.* Wiley, 1998, xvii + 647 pp, $79.95 (P). [ISBN 0-471-16442-9] From the Preface: "We have streamlined some topics and added new sections on theory and on skill-building; we have moved some material into separate sections on modeling." (*First Edition*, TR, February 1994.)

Real Analysis, T(17: 3, 4), P, L. *Fundamentals of Real Analysis.* Sterling K. Berberian. Universitext. Springer-Verlag, 1999, xi + 479 pp, $44.95 (P). [ISBN 0-387-98480-1] Lecture notes from year-long course given at the University of Texas (1985–86). Includes introductory chapters on foundations and basic topology, then detailed treatment of Lebesgue and abstract measure theory leading to function spaces. Includes other topics. Plenty of exercises. KS

Complex Analysis, T(18: 1), P. *Computational Conformal Mapping.* Prem K. Kythe. Birkhäuser Boston, 1998, xv + 462 pp, $69.95. [ISBN 0-8176-3996-9] The theory and computation of conformal mappings of simply or multiply connected regions onto the unit disk and other canonical regions. Applies theory to mathematics, physics, and engineering. PG

Complex Analysis, P. *Complex Geometric Analysis in Pohang.* Eds: Kang-Tae Kim, Steven G. Krantz. Contemp. Math., V. 222. AMS, 1999, vii + 256 pp, $55 (P). [ISBN 0-8218-0957-1] Proceedings of a 1997 conference on several complex variables at Pohang University (South Korea).

Dynamical Systems, P. *Nonlocal Bifurcations.* Yu. Ilyashenko, Weigu Ki. Math. Surv. & Mono., V. 66. AMS, 1999, xiii + 286 pp, $69. [ISBN 0-8218-0497-9] Modern theory of normal forms for local families of vector fields and diffeomorphisms, hyperbolic theory, study of bifurcations on boundaries of Morse–Smale systems. RM

Numerical Analysis, S(17), P. *Numerical Linear Algebra for High-Performance Computers.* Jack J. Dongarra, *et al.* SIAM, 1998, xviii + 342 pp, $37 (P). [ISBN 0-89871-428-1] Surveys the state of the art of solving systems of linear equations and large-scale eigenvalue problems on high-performance (i.e., vector and parallel) computers. A major revision of *Solving Linear Systems on Vector and Shared Memory Computers* (TR, May 1991). AO

Functional Analysis, P. *Banach Algebras '97.* Eds: Ernst Albrecht, Martin Mathieu. Walter de Gruyter, 1998, x + 566 pp, $148.95. [ISBN 3-11-015466-8] Proceedings of a conference held at the University of Tübingen. Research articles, survey articles on problems in automatic continuity and problems related to notions of amenability, and a list of open questions.

Analysis, P, L. *A Primer of Infinitesimal Analysis.* John L. Bell. Cambridge Univ Pr, 1998, xiii + 122 pp, $29.95. [ISBN 0-521-62401-0] Develops basic calculus (single and multivariable) and some physical applications in the context of smooth infinitesimal analysis; includes a chapter on synthetic differential geometry. Theory based on nilpotent infinitesimals (from category theory) rather than nonstandard analysis. AO

Analysis, P. *Wavelets and Their Applications:*

Case Studies. Ed: Mei Kobayashi. SIAM, 1998, xvi + 142 pp, $32 (P). [ISBN 0-89871-416-8] 5 independent essays describe the use of wavelet techniques in mechanical and nuclear engineering, seismology, signal processing, and partial differential equations.

Algebraic Geometry, P. *Higher Homotopy Structures in Topology and Mathematical Physics.* Ed: John McCleary. Contemp. Math., V. 227. AMS, 1999, xii + 321 pp, $69 (P). [ISBN 0-8218-0913-X] Proceedings of a 1996 conference at Vassar College held to honor the 60th birthday of Jim Stasheff.

Geometry, P. *Advances in Discrete and Computational Geometry.* Eds: Bernard Chazelle, Jacob E. Goodman, Richard Pollack. Contemp. Math., V. 223. AMS, 1999, xi + 463 pp, $99 (P). [ISBN 0-8218-0674-2] Proceedings of the 1996 AMS–IMS–SIAM Joint Summer Research Conference "Discrete and Computational Geometry: Ten Years Later" held at Mount Holyoke College.

Algebraic Topology, T(18), P. *Model Categories.* Mark Hovey. Math. Surv. & Mono., V. 63. AMS, 1999, xii + 209 pp, $54. [ISBN 0-8218-1359-5] Much needed comprehensive resource on the relationship between a model category and its homotopy category. Accessible to graduate students with some background in homological algebra. JD

Optimization, T(16–17: 1), P. *Maxima and Minima with Applications: Practical Optimization and Duality.* Wilfred Kaplan. Ser. in Disc. Math. & Optim. Wiley, 1999, x + 284 pp, $74.95. [ISBN 0-471-25289-1] 4 chapters: basic concepts and geometric aspects; problems with side conditions; optimization and mathematical programming; Frenchel–Rockafellar duality theory. Prerequisites: (advanced) calculus and linear algebra. AO

Optimization, T(16–17: 1), P, L. *Integer Programming.* Laurence A. Wolsey. Ser. in Disc. Math. & Optimiz. Wiley, 1998, xviii + 264 pp, $59.95. [ISBN 0-471-28366-5] Incorporates recent developments (e.g., cutting plane theory, heuristic methods). Assumes some knowledge of linear programming and graph theory, but otherwise self-contained. AO

Optimal Control, P. *Geometric Control and Non-Holonomic Mechanics.* V. Jurdjevic, R.W. Sharpe. Conf. Proc., V. 25. AMS, 1998, xi + 239 pp, $49 (P). [ISBN 0-8218-0795-1] Proceedings of a 1996 conference in Mexico City.

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Probability, S, P, L. *The Design Inference: Eliminating Chance Through Small Probabilities.* William A. Dembski. Stud. in Prob., Induction, & Decision Theory. Cambridge Univ Pr, 1998, xvii + 243 pp, $54.95. [ISBN 0-521-62387-1] Not a text but a philosophical tract about when one can infer design behind events of very small probability. Thought provoking, fun to read, full of interesting examples. SN

Stochastic Processes, P. *Ergodicity and Stability of Stochastic Processes.* A.A. Borovkov. Transl: V. Yurinsky. Ser. in Prob. & Stat. Wiley, 1998, xxiii + 585 pp, $175. [ISBN 0-471-97913-9] In-depth treatment of the asymptotic behavior and resulting invariant distributions of Markov chains and some of their generalizations. Includes applications to queueing and communication networks. SN

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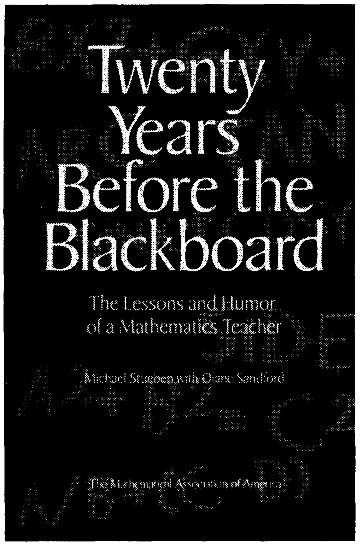

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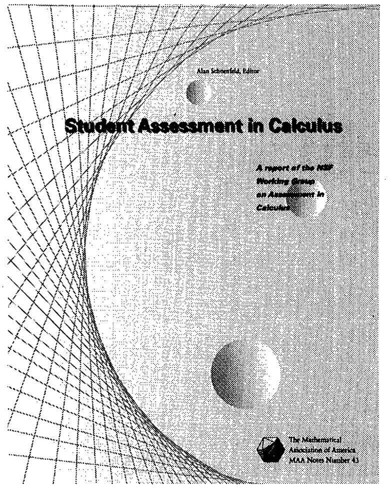

Student Assessment in Calculus

A Report of the NSF Working Group on Assessment in Calculus

ALAN SCHOENFELD, EDITOR
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- summarize major goals of the reform movement and describe the challenges faced by those who are taking a closer look at how students learn;
- illustrate the ways in which calculus projects attempt (via exams, papers, projects, etc.) to find out what their students have learned.

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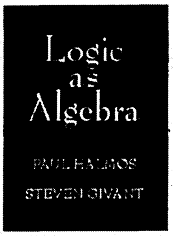

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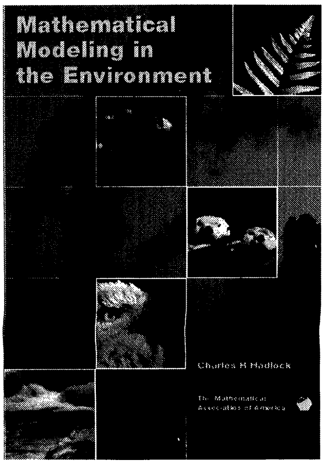

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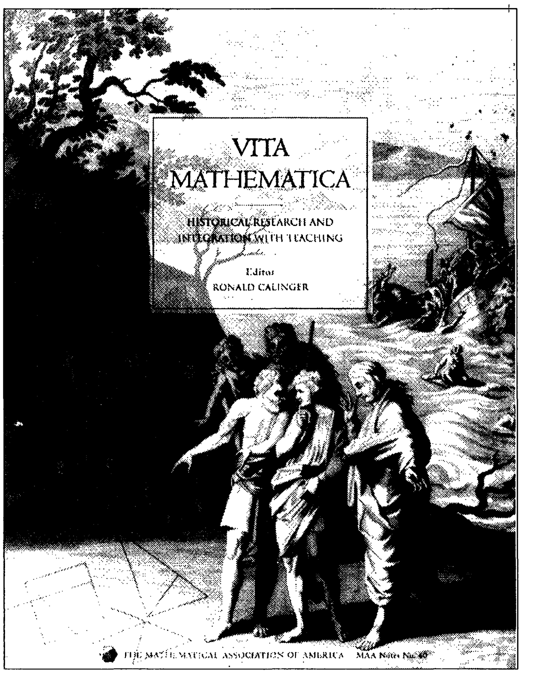
Vita Mathematica

Historical Research and Integration with Teaching

Ronald Calinger, Editor



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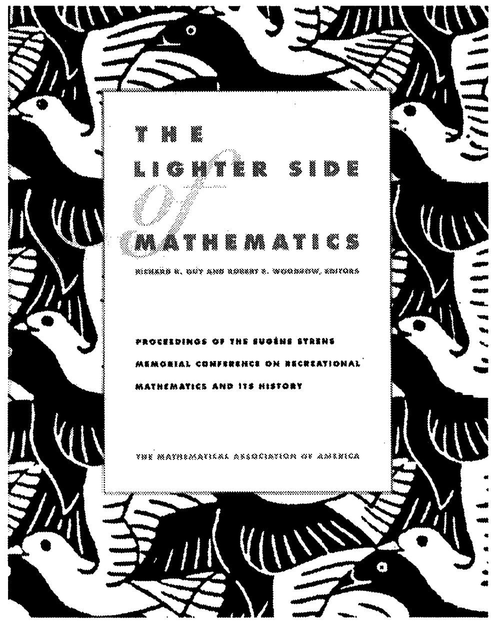

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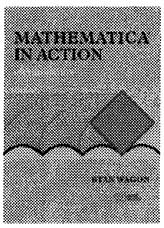

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